B.Sc (3rd Year)

Linear Transformation

(1) Definition: Linear Transformation
(2) Linear operator, Linear functional.

Theorem: If V & W are vector spaces. Then prove that $T: V \longrightarrow W$ is a linear transformation iff $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$.
& Examples.

Theorem 2: Zero Transformations
3: Identity operator.
4. Negative of Linear Trans.
5. Properties of Linear Trans.
6. and Examples

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LINEAR TRANSFORMATION

Definition 1. LINEAR TRANSFORMATION :

If V (F) and W (F) are two vector spaces, then a mapping T from V to W *i.e.*, $T : V \to W$ is said to be a **linear transformation** (or vector space homomorphism or linear mapping) if and only if

(*i*) $T(v + w) = T(v) + T(w) \;\forall\; v, w \in V$

i.e., T takes the sum $v + w$ of V to the sum of the T-images (which is $T(v) + T(w)$) in W.

(*ii*) $T(a\,v) = a\,T(v) \;\forall\; v \in V$ and $a \in F$.

i.e., T takes the scalar multiplication $a\,v$ of V to the scaler multiplication $a\,T(v)$ in W.

Note. (*a*) The property (*i*) is known as Addative Property of T and (*ii*) is known as Homogenous Property of T.

(*b*) A linear transformation is abbreviated as L.T.

Definition 2. Linear Operator : If V (F) is a vector space. Then the linear transformation $T : V \to V$ is called **linear operator** (L.O.)

Definition 3. Linear Functional : If V (F) is a vector space. Then the linear transformation $T : V \to F$ is called **linear functional.**

Theorem 1. If V (F) and W (F) are vector spaces. Then prove that $T : V \to W$ is a linear transformation if and only if

$$T(\alpha v + \beta w) = \alpha\,T(v) + \beta\,T(w) \;\forall\; v, w \in V \text{ and } \alpha, \beta, \in F.$$

Proof. Let $T : V \to W$ be a linear transformation.

Let $v, w \in V$ and $\alpha, \beta, \in F$

$\therefore\; T(\alpha v + \beta w) = T(\alpha v) + T(\beta w)$ [By Addative property of T]

$= \alpha\,T(v) + \beta\,T(w)$ [By Homogenous property of T]

Thus $T(\alpha v + \beta w) = \alpha\,T(v) + \beta\,R(w) \;\forall\; \alpha, \beta \in F$ and $v, w \in V$

Conversely. It is given that

$$T(\alpha v + \beta w) = \alpha\,T(v) + \beta\,T(w) \;\forall\; \alpha, \beta \in F \text{ and } v, w \in V$$

Firstly Take $\alpha = \beta = 1$

we get $T(1.v + 1.w) = 1.\,T(v) + 1.\,T(w)$

$\Rightarrow \quad T(v+w) = T(v) + T(w)$

Secondly Take $\beta = 0$

Then given implies that

$T(\alpha v + 0.w) = \alpha T(v) + 0\,T(w)$

$\Rightarrow \quad T(\alpha v) = \alpha T(v)$

Thus $T: V \to W$ satisfies (i) $T(v+w) = T(v) + T(w)$

(ii) $T(\alpha v) = \alpha T(v)$ for $\alpha \in F, v, w \in V$

$\therefore$ T is a linear Transformation.

Note. (i) We shall use the result of above Theorem to show a given mapping T is a L.T.

(ii) If we want to check whether a given mapping is a L.T. or not then we shall check both properties separately for that mapping.

ILLUSTRATIVE EXAMPLES

Example 1. Show that the following mapping are linear transformations :

(i) $T: V_3(R) \to V_2(R)$ defined by $T(x, y, z) = (x - y + z, 2x)$

(ii) $T: V(R) \to V(R)$ defined by $(x + iy) = x - iy$

where $V(R) = \{x + iy \mid x, y \in R$ and $i = \sqrt{-1}\}$.

Sol. (i) Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2) \in V_3(R)$ and α, β be any two real numbers

$\therefore \quad \alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$\in V_3(R)$

Now $T(\alpha u + \beta v) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$= ((\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2), 2(\alpha x_1 + \beta x_2))$ (by def. of given T)

$= (\alpha x_1 - \alpha y_1 + \alpha z_1 + \beta x_2 - \beta y_2 + \beta z_2, 2\alpha x_1 + 2\beta x_2)$

$= (\alpha(x_1 - y_1 + z_1) + \beta(x_2 - y_2 + z_2), 2\alpha x_1 + 2\beta x_2)$

$= (\alpha(x_1 - y_1 + z_1), 2\alpha x_1) + (\beta(x_2 - y_2 + z_2), 2\beta x_2)$

$= \alpha(x_1 - y_1 + z_1, 2x_1) + \beta(x_2 - y_2 + z_2, 2x_2)$

$= \alpha T(x_1, y_1, z_1) + \beta T(x_2, y_2, z_2)$

$= \alpha T(u) + \beta T(v)$

Hence T is a Linear Transformation.

(ii) Let $u = x_1 + iy_1$ and $v = x_2 + iy_2 \in V(R)$

and $\alpha, \beta \in R$

$\therefore \quad \alpha u + \beta v = \alpha(x_1 + iy_1) + \beta(x_2 + iy_2)$

$= (\alpha x_1 + \beta x_2) + i(\alpha y_1 + \beta y_2)$

$\in V(R)$

Now $T(\alpha u + \beta v) = ((\alpha x_1 + \beta x_2) + i(\alpha y_1 + \beta y_2))$

$= (\alpha x_1 + \beta x_2) - i(\alpha y_1 + \beta y_2)$ (by def. of T)

$= \alpha x_1 - i\alpha y_1 + \beta x_2 - iy_2$

$= \alpha(x_1 - iy_1) + \beta(x_2 - iy_2)$

$= \alpha T(x_1 + iy_1) + \beta T(x_2 + iy_2)$

$= \alpha T(u) + \beta T(v)$

Hence T is a Linear Transformation.

Example 2. Show that the following mappings are not linear transformations :

(i) $T: V_3(R) \to V_2(R)$ defined by $T(x, y, z) = (|y|, 0)$ (G.N.D.U. 2005)

(ii) $T: R^2 \to R^3$ defined by $T(x, y) = (x + 1, 2y, x + y)$ (G.N.D.U. 2005)

(iii) $T: R^2 \to R$ defined by $T(x, y) = xy$

Sol. (i) Let $u = (x_1, y_1, z_1)$ and $V = (x_2, y_2, z_2) \in V_3(R)$

Then $\quad u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$\in V_3(R)$

Now $T(u + v) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (|y_1 + y_2|, 0)$ ...(I)

and $T(u) + T(v) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$= (|y_1|, 0) + (|y_2|, 0)$

$= (|y_1| + |y_2|, 0)$ ...(II)

From (I) & (II) $T(u + v) \neq T(u) + T(v)$

$[\because |y_1 + y_2| = |y_1| + |y_2|$ does not hold for all reals $y_1, y_2]$

Hence T is not a linear transformation.

(ii) Let $\quad u = (x_1, y_1)$ and $v = (x_2, y_2) \in R^2$

Ten $\quad u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$\in R^2$

$\therefore \quad T(u + v) = T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2 + 1, 2(y_1 + y_2), (x_1 + x_2) + (y_1 + y_2))$ ...(I)

$= (x_1 + x_2 + 1, 2y_1 + 2y_2, (x_1 + y_1) + (x_2 + y_2))$

and $T(u)+T(v) = T(x_1, y_1) + T(x_2, y_2)$

$$= (x_1+1, 2y_1, x_1+y_1) + (x_2+1, 2y_2, x_2+y_2)$$

$$= ((x_1+1)+(x_2+1), 2y_1+2y_2, (x_1+y_1)+(x_2+y_2))$$

$$= (x_1+x_2+2, 2y_1+2y_2, (x_1+y_1)+(x_2+y_2)) \quad \text{...(ii)}$$

From (i) & (ii) $T(u+v) \neq T(u)+T(v)$

Hence T is not a Linear Transformation.

Alter. We know if $T : V \to W$ is a L.T.

Then T takes the zero vector of V into zero vector of W.

Here $T(O) = T(0, 0) = (0+1, 2(0), 0+0)$

$$= (1, 0, 0)$$

$$\neq O$$

Hence T is not a Linear transformation.

(iii) Let $u = (x_1, y_1)$ and $v = (x_2, y_2) \in R^2$

Then $u+v = (x_1, y_1) + (x_2, y_2)$

$$= (x_1+x_2, y_1+y_2) \in R^2$$

$\therefore$ $T(u+v) = T(x_1+x_2, y_1+y_2)$

$$= (x_1+x_2)(y_1+y_2)$$

$$= x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 \quad \text{...(i)}$$

and $T(u)+T(v) = T(x_1, y_1) + T(x_2, y_2)$

$$= x_1y_1 + x_2y_2 \quad \text{...(ii)}$$

From (i) & (ii)

$T(u+v) \neq T(u)+T(v)$

Hence T is not a Linear Transformation.

Example 3. Let P be a vector space of all polynomials over R, show that the mapping

$$T : P \to R \text{ defined by } T(q(x)) = \int_a^b q_1(t)\, q(t)\, dt \quad \text{where } q_1(t) \text{ is any fixed}$$

polynomial in P, is a linear functional on P.

Sol. Let $f(x)$ and $g(x)$ be two polynomials over R.

Let $\alpha, \beta \in R$, Then $\alpha f(x) + \beta g(x) \in P(R)$ [$\because$ P is a vector space]

$$\therefore \quad T(\alpha f(x) + \beta g(x)) = \int_a^b q_1(t)[\alpha f(t) + \beta g(t)]\, dt \quad \text{[by def. of T]}$$

$$= \int_a^b [\alpha\, q_1(t) f(t) + \beta\, q_1(t) g(t)]\, dt$$

$$= \int_a^b \alpha q_1(t) f(t)\, dt + \int_a^b \beta q_1(t) g(t)\, dt$$

$$= \alpha \int_a^b q_1(t) f(t)\, dt + \beta \int_a^b q_1(t) g(t)\, dt$$

$$= \alpha\, T(f(x)) + \beta\, T(g(x))$$

Hence T is linear functional.

EXERCISE 4 (a)

1. Show that the following mappings are linear transformations :
   (i) $T : R^3 \to R^2$ defined by $T(x, y, z) = (x - y, x + z)$
   (ii) $T : R^2 \to R^3$ defined by $T(x, y) = (x + y, x - y, y)$
   (iii) $T : R^3 \to R$ defined by $T(x, y, z) = x + 3y - 4z$
   (iv) $T : R^2 \to R^2$ defined by $T(x, y) = (0, -x)$
2. Show that the following mappings are not linear transformations :
   (i) $T : R^2 \to R$ defined by $T(x, y) = |2x - 3y|$
   (ii) $T : V(C) \to V(C)$ defined by $T(x + iy) = z$
   where $V(C) = \{x + iy \mid x, y \in R \text{ and } i = \sqrt{-1}\}$
   (iii) $T : V(C) \to V(R)$ defined by $T(x + iy) = (x^3 + y^3)^{\frac{1}{3}}$
3. Let F be a field, which of the following are linear functionals ?
   (i) $T : F^3 \to F$ defined by $T(a, b, c) = a + 3b - 2c$
   (ii) $T : F^3 \to F$ defined by $T(a, b, c) = (b - c)^2$
   (iii) $T : F^3 \to F$ defined by $T(a, b, c) = a + 1$ where 1 is unity of F.

   [Ans. (i) Linear functional (ii) Not a linear functional (iii) Not a linear functional**]**
4. Let $T : R \to R^2$ defined by $T(x) = (2x, 3x)$.
   Prove that T is a Linear Transformation.
5. Let $T : R^3 \to R^2$ defined by
   $T(x, y, z) = (3x - 2y + z, x - 3y - 2z)$ for all x, y, z in R.
   Prove that T is a Linear Transformation.

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6. If V (F) and W (F) be two vector spaces and T_1, T_2 are linear transformations from V into W. Prove that the mapping T defined by $T(v) = a\,T_1(v) + T_2(v)$, $v \in V$, $a \in F$ is a linear transformation from V into W.

7. Let F be a field and V be the space of polynomial functions from F into F defined by

$$h(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n \,;\, c_i\text{'s} \in F$$

Let $D(h(x)) = c_1 + 2c_2 x + \dots + n c_n x^{n-1}$

Prove that $D : V \to V$ is a linear operator.

8. Let V (R) be a vector space of integrable functions on R. Prove that $T : V \to R$ defined by

$$T(f) = \int_c^d f(x)\,dx,\ f \in V \,;\ c, d \in R \text{ is a linear functional.}$$

9. Let R be the field or reals and V be the space of all functions from R into R which are continuous.

Define T by $(Tf)x = \int_0^x f(t)\,dt$

Prove T is a linear operator from V into V

10. Let V (F) be a vector space of all $m \times n$ matrices over a field F and P and Q are two fixed matrices of order $m \times m$ and $n \times n$ respectively over the same field F. Prove that a mapping $T : V \to V$ such that

$$T(A) = PAQ \ \forall\ A \in V \text{ is a linear operator.}$$

11. If V(F) be a vector space of all $n \times n$ matrices and let P be a fixed $n \times n$ matrix. Then, prove that the mapping T defined as

(i) $T(A) = AP - PA$ (ii) $T(A) = AP + PA$

are linear transformation from V into V.

12. Let P be the set of all polynomials in y over the field of reals and let

$$F(y) = \sum_{i=0}^{n-1} a_i y^i \in P$$

Now a transformation $T : P \to P$ is defined as

$$T(F(y)) = \sum_{i=0}^{n-1} a_i \frac{y^{i+1}}{i+1}$$

Prove that T is a linear operator.

13. If V and W are two vector spaces over the same filed F.

Show $T : V \to W$ is a L.T

iff $T(\lambda v_1 + v_2) = \lambda T(v_1) + T(v_2)\ \forall\ v_1, v_2 \in V$ and $\lambda \in F$

14. Let V be vector space of dim 3 over F and $\{v_1, v_2, v_3\}$ be a basis of V. If any $v \in V$ be represented as $v = a_1 v_1 + a_2 v_2 + a_3 v_3$; $a_1, a_2, a_3 \in F$.

Determine, which of following are linear functional on T

(i) $f(v) = a_1 + 2a_2 - a_3$

(ii) $f(v) = (a_1 - a_2)^2$

(iii) $f(v) = a_1 - 1$ where 1 is unity of F.

[Ans. (i) Yes, (ii) No, (iii) No]

Theorem 2. Zero Transformation (or operator)

If V(F) and W (F) are vector spaces, then a mapping $T : V \to W$ defined as

$$T(x) = 0\ \forall\ x \in V$$

is a linear transformation (operator). Prove it.

Proof. Let $x, y \in V$ and $\alpha, \beta \in F$, so that $T(x) = 0$ and $T(y) = 0$ ...(I)

$\Rightarrow\ \alpha x + \beta y \in V$ [$\because$ V is vector space]

$\therefore\ T(\alpha x + \beta y) = 0$ [by def. of T]

$= \alpha . 0 + \beta . 0$

$= \alpha T(x) + \beta T(y)$ [$\because$ of (I)

Hence T is a linear transformation.

Note : It is called Zero Transformation denoted by O

i.e. $O(x) = 0\ \forall\ x \in V$

Theorem 3. Identity Operator : If V (F) is a vector space, then the mapping T defined as $T(x) = x\ \forall\ x \in V$ is a linear operator on V

Proof. Let $x, y \in V$ and $\alpha, \beta \in F$, so that $T(x) = x$ and $T(y) = y$ ...(I)

$\Rightarrow\ \alpha x + \beta y \in V$ [$\because$ V is a vector space]

$\therefore\ T(\alpha x + \beta y) = \alpha x + \beta y$ [by def. of T]

$= \alpha T(x) + \beta T(y)$ [$\because$ of (I)]

Hence T is a linear operator.

Note. It is called Identity operator denoted by I

i.e. $I(x) = x\ \forall\ x \in V$

Theorem 4. Negative of a linear Transformation. If V (F) and W (F) are vector spaces and $T : V \to W$ is a linear transformation. Then show that mapping $-T : V \to W$ defined as $(-T)(x) = -[T(x)]\ \forall x \in V$ is a linear transformation.

Proof. Given $T : V \to W$ is a linear transformation

$\therefore$ $T(x) \in W$ for $x \in V$

$\Rightarrow$ $-T(x) \in W$ [$\because$ W is a vector space]

Let $\alpha, \beta \in F$ and $x, y \in V$

$\Rightarrow$ $\alpha x + \beta y \in V$ [$\because$ V is a vector space]

$$
\begin{aligned}
\therefore\ (-T)(\alpha x + \beta y) &= -[T(\alpha x + \beta y)] && \text{(by def. of T)}\\
&= -[\alpha T(x) + \beta T(y)] && (\because \text{T is linear})\\
&= -\alpha T(x) - \beta T(y)\\
&= \alpha(-T(x)) + \beta(-T(y))\\
&= \alpha[-T(x)] + \beta[(-T)(y)]
\end{aligned}
$$

$\Rightarrow$ $-T : V \to W$ is a L.T.

Theorem 5. *(Properties of linear transformations)*

If $T : V \to W$ is a linear transformation from V (F) to W (F). Then

(*i*) $T(O) = O$, where O on left hand $\in$ V and O on right hand $\in$ W

(*ii*) $T(-x) = -T(x)\ \forall x \in V$

(*iii*) $T(x - y) = T(x) - T(y)\ \forall x, y \in V$

(*iv*) $T(px) = pT(x)\ \forall x \in V, p \in I$

Proof. (*i*) Let $T(x) = w$ for $x \in V$, $w \in W$ ...(*i*)

Then

$$
\begin{aligned}
T(x) &= T(x + O)\\
&= T(x) + T(O) && [\because \text{T is a L.T.}]\\
\Rightarrow\quad w &= w + T(O) && [\because \text{of } (i)]\\
\Rightarrow\quad w + O &= w + T(O) && [\text{by cancellation Law}]\\
\Rightarrow\quad O &= T(O)\\
\Rightarrow\quad T(O) &= O.
\end{aligned}
$$

(*ii*)

$$
\begin{aligned}
T(-x) &= T((-1)x)\\
&= (-1)T(x) && [\because \text{T is a L.T.}]\\
&= -T(x)
\end{aligned}
$$

(*iii*)

$$
\begin{aligned}
T(x - y) &= T(x + (-y))\\
&= T(x) + T(-y) && [\because \text{T is a L.T.}]\\
&= T(x) - T(y) && [\text{because of } (ii)]
\end{aligned}
$$

(*iv*) We shall prove this result by principle of induction on p

Case I. When p is a + ve integer

For $p = 1$, $T(1 \cdot x) = T(x) = 1 \cdot T(x)$ so that result is true for $p = 1$.

Suppose result is true for $p = m$, m is a + ve integer

i.e. $T(mx) = mT(x)$

Now

$$
\begin{aligned}
T((m+1)x) &= T(mx + x)\\
&= T(mx) + T(x)\\
&= mT(x) + T(x) && [\because T(mx) = mT(x)]\\
&= (m+1)T(x)
\end{aligned}
$$

$\therefore$ Result is true for $p = m + 1$

Hence by induction, $T(px) = pT(x)$ for all + ve integers p.

Case II. When $p = 0$, then $T(0 \cdot x) = T(0) = 0 = 0 \cdot T(x)$

$\therefore$ Result is true for $p = 0$.

Case III. When p is a negative integer, let $p = -q$ where q is a + ve integer

$$
\begin{aligned}
T(px) &= T((-q)x) = T(q(-x)) = qT(-x) = q(-T(x))\\
&= (-q)T(x) = pT(x)
\end{aligned}
$$

$\therefore$ Result is true for negative integers p.

Hence the result is true for all integers p.

Theorem 6. Let V and W be vector spaces over the same field F.

Let $B_1 = \{v_1, v_2, \ldots\ldots, v_n\}$ be a basis of V and let $w_1, w_2, \ldots\ldots, w_n$ be any vectors in W. Prove then there exists a unique linear mapping $T : V \to W$ such that

$$T(v_i) = w_i,\ 1 \le i \le n.$$

Proof. We shall prove the theorem in three steps.

(*i*) Define a mapping $T : V \to W$ such that $T(v_i) = w_i, 1 \le i \le n$

(*ii*) To show T is linear

(*iii*) To show T in Unique

Step (*i*) Existence of T

Let $v \in V$. Since $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of V, so there exists unique scalars $a_1, a_2, \ldots, a_n \in F$ s.t.

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \qquad \ldots(1)$$

Let $T : V \to W$ be defined as

$$T(v) = a_1 w_1 + a_2 w_2 + \ldots + a_n w_n$$

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$\because$ a_i's $(1 \le i \le n)$ are unique, the mapping T from V to W is well defined.

Now each $v_i \in V$ can be expressed as a linear combination of vectors of basis B

i.e. $v_i = 0\,v_1 + 0\,v_2 + \ldots + 1 \cdot v_i + \ldots + 0\,v_n$

$\therefore$ $T(v_i) = 0\,w_1 + 0\,w_2 + \ldots + 1 \cdot w_i + \ldots + 0\,w_n$ [Because of (1)]

i.e. $T(v_i) = w_i$, for $i, 1, \ldots\ldots, n$.

Step (ii) To show that T is linear

Let $u, v \in V$ and $\alpha, \beta \in F$ where u and v can be written as

$$u = \sum_{i=1}^{n} b_i v_i \text{ and } v = \sum_{i=1}^{n} c_i v_i \text{ for } b_i\text{'s}, c_i\text{'s} \in F$$

[$\because$ $B = \{v_1, v_2, \ldots\ldots, v_n\}$ is a basis of V]

$\Rightarrow$ $T(u) = \sum_{i=1}^{n} b_i w_i$ and $T(v) = \sum_{i=1}^{n} c_i w_i$

$\therefore$ $\alpha u + \beta v \in V$ [$\because$ V is a vector space]

Now $T(\alpha u + \beta v) = T\left(\alpha \sum_{i=1}^{n} b_i v_i + \beta \sum_{i=1}^{n} c_i v_i\right)$

$$= T\left(\sum_{i=1}^{n} (\alpha b_i) v_i + \sum_{i=1}^{n} (\beta c_i) v_i\right) = T\left(\sum_{i=1}^{n} (\alpha b_i + \beta c_i) v_i\right)$$

$$= \sum_{i=1}^{n} (\alpha b_i + \beta c_i) w_i \quad \text{[by def. of T]}$$

$$= \sum_{i=1}^{n} (\alpha b_i)(w_i) + \sum_{i=1}^{n} (\beta c_i)(w_i)$$

$$= \alpha \sum_{i=1}^{n} b_i w_i + \beta \sum_{i=1}^{n} c_i w_i$$

$$= \alpha\, T(u) + \beta \cdot T(v)$$

Hence T is a linear mapping.

Step (iii) To show that T is unique.

Let $S : V \to W$ be another linear mapping such that

$S(v_i) = w_i$, $i = 1, 2, \ldots n$

If $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \in V(F)$

then $S(v)$ $= S(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n)$

$= a_1 S(v_1) + a_2 S(v_2) + \ldots + a_n S(v_n)$ [$\because$ S is linear]

$= a_1 w_1 + a_2 w_2 + \ldots + a_n w_n$

$= T(v)$ [by def. of T]

$S(v) = T(v)\ \forall\, v \in V$ so that $S = T$.

Thus T is unique.

Theorem 7. Let $T : V(F) \to W(F)$ be a linear transformation.

Prove that (i) If the vectors $v_1, v_2, \ldots\ldots, v_n \in V$ are L.D. over F

Then $T(v_1), T(v_2), \ldots, T(v_n) \in W$ are also L.D. over F

(G.N.D.U. 2008)

(ii) If the vectors $v_1, v_2, \ldots\ldots, v_n \in V$ are such that their images $T(v_1), T(v_2), \ldots, T(v_n) \in W$ are L.I. over F. Then $v_1, v_2, \ldots\ldots, v_n \in V$ are L.I. over F

Proof. (i) Since $v_1, v_2, \ldots, v_n \in V$ are L.D. over F

$\exists$ scalars $a_1, a_2, \ldots, a_n \in F$, not all zero such that

$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$

$\Rightarrow$ $T(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) = T(0)$

$\Rightarrow$ $a_1 T(v_1) + a_2 T(v_2) + \ldots + a_n T(v_n) = 0$ [$\because$ T is linear]

Thus $T(v_1), T(v_2), \ldots, T(v_n) \in W$ are L.D.

Hence the result.

(ii) Suppose, $\exists$ scalars $a_1, a_2, \ldots, a_n \in F$ such that

$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ for v_i's $\in V\ (1 \le i \le n)$

Then $T(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) = T(0)$

$\Rightarrow$ $a_1 T(v_1) + a_2 T(v_2) + \ldots + a_n T(v_n) = 0$ [$\because$ T is linear]

Given $T(v_1), T(v_2), \ldots T(v_n)$ are L.I. over F

$\therefore$ $a_1 = a_2 = \ldots = a_n = 0$

So that $a_1 v_1 + a_2 v_2 + \ldots a_n v_n = 0 \Rightarrow a_1 = a_2 = \ldots = a_n = 0$

$\therefore$ $v_1, v_2, \ldots v_n$ are L.I.

Hence the result.

ILLUSTRATIVE EXAMPLES

Example 1. Describe explicitly the linear transformation $T : R^2 \to R^2$ such that $T(e_1) = (\alpha, \beta)$ and $T(e_2) = (a, b)$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are unit vectors.

Sol. Let $(x, y) \in R^2$

We are to find $T(x, y)$ under the given conditions $T(e_1) = (a, \beta)$ and $T(e_2) = (a, b)$

We know that $\{e_1, e_2\}$ is a basis set for R^2

$\therefore$ Any vector $(x, y) \in R^2$ can be expressed as a linear combination of the elts of basis set

Clearly $(x, y) = x(1, 0) + y(0, 1)$

$= x e_1 + y e_2$

$\therefore$ $T(x, y) = T(x e_1 + y e_2)$

$= x\,T(e_1) + y\,T(e_2)$ [$\because$ T is linear]

$= x(a, \beta) + y(a, b)$

$= (a x, \beta x) + (a y, b y)$

$= (a x + a y, \beta x + b y)$

so that $T(x, y) = (a x + a y, \beta x + b y)$ is the required linear transformation.

Example 2. Find $T(x, y)$ where $T : R^2 \to R^3$ is defined as

$T(2, -5) = (-1, 2, 3)$ and $T(3, 4) = (0, 1, 5)$

(G.N.D.U. 2008, 20..)

Sol. Firstly we shall show that given vectors $(2, -5)$ and $(3, 4)$ of domain of T form a basis for R^2 (= domain of T).

(a) To show $(2, -5)$ and $(3, 4)$ are L.I.

Consider $\alpha(2, -5) + \beta(3, 4) = 0$ for α, β any scalars

$\Rightarrow$ $(2\alpha, -5\alpha) + (3\beta, 4\beta) = (0, 0)$

$\Rightarrow$ $(2\alpha + 3\beta, -5\alpha + 4\beta) = (0, 0)$

$\therefore$ $2\alpha + 3\beta = 0$ and $-5\alpha + 4\beta = 0$

$\Rightarrow$ $\alpha = \beta = 0$

Thus $(2, -5)$ and $(3, 4)$ are L.I.

(b) To show $(2, -5)$ and $(3, 4)$ span R^2

Let $(x, y) \in R^2$

Let $(x, y) = \alpha(2, -5) + \beta(3, 4)$

$= (2\alpha + 3\beta, -5\alpha + 4\beta)$

$\therefore$ $2\alpha + 3\beta = x$ and $-5\alpha + 4\beta = y$

$\Rightarrow$ $\alpha = \dfrac{4x - 3y}{7}$ and $\beta = \dfrac{5x + 2y}{23}$

Thus $(x, y) = \dfrac{4x - 3y}{23}(2, -5) + \dfrac{5x + 2y}{23}(3, 4)$ ...(i)

Hence $(2, 5)$ and $(3, 4)$ span R^2

$T(x, y) = T\left(\dfrac{4x - 3y}{23}(2, -5) + \dfrac{5x + 2y}{23}(3, 4)\right)$ [by (i)]

$= \dfrac{4x - 3y}{23}\,T(2, -5) + \dfrac{5x + 2y}{23}\,T(3, 4)$ [$\because$ T is linear]

$= \dfrac{4x - 3y}{23}(-1, 2, 3) + \dfrac{5x + 2y}{23}(0, 1, 5)$ [because of given]

$= \left(\dfrac{-4x + 3y}{23}, \dfrac{8x - 6y}{23}, \dfrac{12x - 9y}{23}\right) + \left(0, \dfrac{5x + 2y}{23}, \dfrac{25x + 10y}{23}\right)$

$= \left(\dfrac{-4x + 3y}{23}, \dfrac{13x - 4y}{23}, \dfrac{37x + y}{23}\right)$

so that $T(x, y) = \left(\dfrac{-4x + 3y}{23}, \dfrac{13x - 4y}{23}, \dfrac{37x + y}{23}\right)$ is the required linear transformation.

Example 3. Find a linear transformation $T : R^2 \to R^2$ such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$.

Prove that T maps the square with vertices $(0, 0)$ $(1, 0)$, $(1, 1)$ and $(0, 1)$ into a parallelogram.

Sol. Let $(x, y) \in R^2$. Firstly we shall find $T(x, y)$ under given conditions

$T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$

We know $\{(1, 0), (0, 1)\}$ is a basis set for R^2 (Prove here)

$\therefore$ any vector $(x, y) \in R^2$ can be expressed as a linear combination of elts of this basis.

And $(x, y) = x(1, 0) + y(0, 1)$

$\Rightarrow$ $T(x, y) = T(x(1, 0) + y(0, 1))$

$= x\,T(1, 0) + y\,T(0, 1)$ [$\because$ T is a L.T.]

$= x(1, 1) + y(-1, 2)$ [because of given]

$= (x - y, x + 2y)$

so that $T(x, y) = (x - y, x + 2y)$ is the required L.T. ...(i)

IInd Part. Let the vertices of square be P, Q, R, S resp. and their T-images be A, B, C, D resp.

$\therefore$ $A = T(P) = T(0, 0) = (0, 0)$ [$\because$ Put $x = 0, y = 0$ in (i)]

$B = T(Q) = T(1, 0) = (1, 1)$ [$\because$ Put $x = 1, y = 0$ in (i)]

$C = T(R) = T(1, 1) = (0, 3)$ [$\because$ Put $x = 1, y = 1$ in (i)]

$D = T(S) = T(0, 1) = (-1, 2)$ [$\because$ Put $x = 0, y = 1$ in (i)]

Mid point of [AC] $=\left(\frac{0+0}{2}, \frac{0+3}{2}\right)=\left(0, \frac{3}{2}\right)$

Mid point of [BD] $=\left(\frac{1+(-1)}{2}, \frac{1+2}{2}\right)=\left(0, \frac{3}{2}\right)$

$\therefore$ Mid point of [AC] = Mid point of [BD]

Thus ABCD is a parallelogram.

Hence T maps square PQRS into a parallelogram ABCD.

Example 4. Find a L.T which transforms $(3, -1, -2)\ (1, 1, 0)\ (-2, 0, 2)$ in R^3 to twice the elementary vectors i.e. $2e_1, 2e_2, 2e_3$ in R^3.

(Here e_1, e_2, e_3 are elementary vectors)

Sol. Let $T: R^3 \to R^3$ be a L.T such that

$T(3, -1, -2)=2e_1,\ T(1, 1, 0)=2e_2,\ T(-2, 0, 2)=2e_3$ (Using given)

Firstly Show $B=\{(3, -1, -2), (1, 1, 0), (-2, 0, 2)\}$ form a basis of R^3

To prove L.I Consider

$\alpha(3, -1, -2)+\beta(1, 1, 0)+\gamma(-2, 0, 2)=(0, 0, 0)$

$\Rightarrow (3\alpha+\beta-2\gamma, -\alpha+\beta, -2\alpha+2\gamma)=(0, 0, 0)$

$\Rightarrow \alpha=\beta=\gamma=0$

$\therefore$ B is L.I set

To prove $L(B)=R^3$

Let $(x, y, z) \in R^3$ be arbitrary element

Take $(x, y, z)=a(3, -1, -2)+b(1, 1, 0)+c(-2, 0, 2)$

$=(3a+b-2c, -a+b, -2a+2c)$

$\Rightarrow 3a+b-2c=x, -a+b=y, -2a+2c=z$

Solving, we get

$$a=\frac{1}{2}(x-y+z),\ b=\frac{1}{2}(x+y+z),\ c=\frac{1}{2}(x-y+2z)$$

$$\therefore (x, y, z)=\frac{1}{2}(x-y+z)(3, -1, -2)+\frac{1}{2}(x+y+z)(1, 1, 0)+\frac{1}{2}(x-y+2z)(-2, 0, 2)$$

i.e. every elt of R^3 can be written as L.C. of elements of B

$\Rightarrow$ B is a basis of R^3

so that

$$T(x, y, z)=T\left(\frac{1}{2}(x-y+z)(3, -1, -2)+\frac{1}{2}(x+y+z)(1, 1, 0)+\frac{1}{2}(x-y+2z)(-2, 0, 2)\right)$$

$$=\frac{1}{2}(x-y+z)T(3, -1, -2)+\frac{1}{2}(x+y+z)T(1, 1, 0)+\frac{1}{2}(x-y+2z)T(-2, 0, 2)$$

$$=\frac{1}{2}(x-y+z)2e_1+\frac{1}{2}(x+y+z)2e_2+\frac{1}{2}(x-y+2z)2e_3$$

(Using given)

$=(x-y+z)(1, 0, 0)+(x+y+z)(0, 1, 0)+(x-y+2z)(0, 0, 1)$

$=(x-y+z, x+y+z, x-y+2z)$

which is required L.T.

EXERCISE 4 (*b*)

1. Find a linear transformation $T: R^2 \to R^2$ such that $T(1, 2)=(3, 4)$ and $T(0, 1)=(0, 0)$.

   (G.N.D.U. 2012)

   [Ans. $T(x, y)=(3x, 4x)$]

2. Describe explicitly the linear transformation $T: R^2 \to R^2$ such that $T(2, 3)=(4, 5)$ and $T(1, 0)=(0, 0)$.

   [Ans. $T(x, y)=\left(\frac{4y}{3}, \frac{5y}{3}\right)$]

3. Find $T(x, y)$ where $T: R^2 \to R^3$ is defined as $T(1, 2)=(3, -1, 5)$ and $T(0, 1)=(2, 1, -1)$.

   (G.N.D.U. 2011)

   [Ans. $T(x, y)=(-x+2y, -3x+y, 7x-y)$]

4. Find $T(a, b, c)$ where $T: R^3 \to R$ is defined by
   (i) $T(1, 1, 1)=3, T(1, 1, 0)=-4, T(1, 0, 0)=2$.
   (ii) $T(1, 1, 1)=3, T(0, 1, -2)=1, T(0, 0, 1)=-2$.

   [Ans. (i) $T(x, y, z)=2x-6y+7z$, (ii) $T(x, y, z)=8x-3y-2z$]

5. A linear transformation $T: R^3 \to R^3$ is defined by
   $T(e_1)=e_1+e_2+e_3,\ T(e_2)=e_2+e_3,\ T(e_3)=e_2-e_3$
   where e_1, e_2, e_3 are unit vectors of R^3.