Example 1. Find the eigen values and corresponding eigen vectors of
$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

(H.

Sol. Let $$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Now $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\therefore \quad A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$\therefore$ characteristic equation of matrix A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

or $(1-\lambda)(4-\lambda) - (2)(2) = 0$

or $\lambda^2 - 5\lambda + 4 - 4 = 0$ or $\lambda^2 - 5\lambda = 0$

or $\lambda(\lambda - 5) = 0$

$\therefore$ $\lambda = 0, 5$ are the eigen values of A.

The eigen vectors $X = \begin{pmatrix} x \\ y \end{pmatrix} \neq O$ corresponding to the eigen values $\lambda = 0$ i

$AX = \lambda X$ or $(A - O)X = O$ or $AX = O$

$$\therefore \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ by } R_2 \to R_2 - R_1$$

Now the coefficient matrix of these equation is of rank 1. Therefore, these equations have only 2 – 1 = 1 L.I. solution. Thus there is only one L.I. eigen vector corresponding to the eigen value 0. These equations can be written as

$$x + 2y = 0 \quad \Rightarrow \quad x = -2y$$

Take $\quad y = 1$

$\therefore \quad x = -2$

$X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 0.

The eigen vector $X = \begin{pmatrix} x \\ y \end{pmatrix} \neq O$ corresponding to the eigen value $\lambda = 5$ is given by

$$AX = 5X \text{ or } (A - 5\,I)\,X = O$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore \quad 2x - y = 0$, or $x = \dfrac{1}{2}y$

Take $y = 2$, $\therefore \quad x = 1$

$\therefore \quad x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is eigen vector corresponding to eigen value 5.

Example 2. Determine eigen values and the corresponding eigen-vectors for the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

(H.P.U. 2013)

Sol. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}$$

∴ the characteristic equation of A is $|A - \lambda I| = 0$

$$\text{or} \quad \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\text{or} \quad (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2-\lambda \\ 2 & 2 \end{vmatrix} = 0$$

$$\text{or} \quad (1-\lambda)[(2-\lambda)(3-\lambda) - 2] - [2 - 2(2-\lambda)] = 0$$

$$\text{or} \quad (1-\lambda)(\lambda^2 - 5\lambda + 4) - (2 - 4 + 2\lambda) = 0$$

$$\text{or} \quad -\lambda^3 + 6\lambda^2 - 9\lambda + 4 + 2 - 2\lambda = 0$$

$$\text{or} \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Put $\lambda = 1$ in (1).

∴ $1 - 6 + 11 - 6 = 0$ or $0 = 0$

∴ $\lambda = 1$ is a root of (1).

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \underline{|0} \end{array}$$

∴ remaining roots of (1) are given by

$$\lambda^2 - 5\lambda + 6 = 0 \quad \text{or} \quad (\lambda - 2)(\lambda - 3) = 0$$

∴ $\lambda = 2, 3$

∴ roots of (1) are $\lambda = 1, 2, 3$, which are eigen values of given matrix A.

The eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq O$ corresponding to eigen value $\lambda = 1$ is given by

$$AX = \lambda X \quad \text{or} \quad (A - I)X = O$$

$$\text{or} \quad \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \leftrightarrow R_2$$

or $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, by $R_3 \to R_3 - 2R_1$

$\therefore \quad x + y + z = 0$

and $-z = 0$ *i.e.* $z = 0$

$\therefore x + y = 0$ *i.e.* $x = -y$

Take $y = 1$, $\therefore x = -1$

$\therefore X = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is an eigen vector of A.

The eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq O$ correspondings to eigen value $\lambda = 2$ is given by

$AX = \lambda X$ or $(A - 2I)X = 0$

or $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, by $R_1 \leftrightarrow R_2$

or $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, by $R_2 \to R_2 + R_1, R_3 \to R_3 - 2R_1$

or $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, by $R_1 \leftrightarrow R_2$

$\therefore x + z = 0$

and $2y - z = 0$

$\therefore x = -z,\ y = \dfrac{z}{2}$

Take $z = 2$, $\therefore x = -2, y = 1$

$$\therefore X = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \text{ is an eigen vector of A.}$$

The eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq O$ correspondings to eigen value $\lambda = 3$ is given by

$$AX = \lambda X \quad \text{or} \quad (A - 3\,I)\,X = 0$$

$$\text{or} \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \leftrightarrow R_2$$

$$\text{or} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - 2R_1$$

$$\text{or} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + 2R_2$$

$\therefore\ x - y + z = 0$

and $-2y + z = 0$

$\therefore\ y = \frac{z}{2},\ x = -\frac{z}{2}$

Take $z = -2$, $\therefore\ x = 1,\ y = -1$

$$\therefore X = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \text{ is an eigen vector of A.}$$

Example 3. Determine the characteristic roots and characteristic vectors of the matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

(G.N.D.U. 2008, 2009, 201

Scanned with CamScanner

Sol. Let

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$\therefore$ characteristic equation of matrix A is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

or $(8-\lambda)\{(7-\lambda)(3-\lambda) - 16\} + 6\{-6(3-\lambda) + 8\} + 2\{24 - 2(7-\lambda)\} = 0$

or $(8-\lambda)\{\lambda^2 - 10\lambda + 5\} + 6\{6\lambda - 10\} + 2\{2\lambda + 10\} = 0$

or $-\lambda^3 + 18\lambda^2 - 85\lambda + 40 + 36\lambda - 60 + 4\lambda + 20 = 0$

or $-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

or $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\therefore$ $\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\therefore$ $\lambda(\lambda - 3)(\lambda - 15) = 0$

$\therefore$ $\lambda = 0, 3, 15$ are the characteristic roots of A.

The eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq O$ corresponding to the eigen value $\lambda = 0$ is given by

$AX = 0X$

$\Rightarrow$ $(A - O)X = O$

$\Rightarrow$ $AX = O$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \to R_3$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 + 3R_1,\ R_3 \to R_3 - 4R_1$$

$$\Rightarrow \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + 2R_2$$

Now the coefficient matrix of these equations is of rank 2. Therefore, these equations have only 3 – 2 = 1 L.I. solution. Thus there is only one L.I. eigen vector corresponding to the eigen value 0. These equations can be written as

$$2x - 4y + 3z = 0$$

$$-5y + 5z = 0 \qquad \Rightarrow y = z$$

Take $\qquad y = z = 1$

$$\therefore \qquad 2x - 4 + 3 = 0 \qquad \Rightarrow x = \frac{1}{2}$$

$$X = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$ is an eigen vector of A corresponding to the eigen value 0.

If c_1 is any non-zero scalar, then $c_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ is also an eigen vector of A corresponding to the eigen value 0.

The eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq O$ corresponding to the eigen value $\lambda = 3$ is given by

$$AX = 3X \text{ or } (A - 3I)X = O$$

$$\text{or} \quad \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \to R_1 + R_2$$

$$\text{or} \quad \begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - 6R_1,\ R_3 \to R_3 + 2R_1$$

$$\text{or}\quad \begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + \frac{1}{2} R_2$$

The coefficient matrix of these equations is of rank 2. Therefore these equation have $3 - 2 = 1$ L.I. solution. These equations can be written as

$$-x - 2y - 2z = 0$$

$$16y + 8z = 0 \qquad \Rightarrow \quad y = -\frac{1}{2} z$$

Take $\quad z = 4, \qquad \therefore \quad y = -2$

$\therefore \quad -x + 4 - 8 = 0 \qquad \Rightarrow \quad x = -4$

$\therefore \quad X = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 3. Every non-zero multiple of $\begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 3.

The eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq O$ corresponding to the given value $\lambda = 15$ is given by

$AX = 15X$ or $(A - 15I)X = O$

$$\text{or}\quad \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or}\quad \begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \to R_1 - R_2$$

$$\text{or}\quad \begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - 6R_1,\ R_3 \to R_3 + 2R_1$$

The coefficient matrix of these equations is of rank 2. Therefore these equations have $3 - 2 = 1$ L.I. solution. These equations can be written as

$$-x + 2y + 6z = 0$$

$$-20y - 40z = 0 \qquad \Rightarrow \quad y = -2z$$

Take $z = 1, \qquad \therefore \quad y = -2$

$\therefore \quad -x-4+6=0 \qquad \Rightarrow x=2$

$\therefore \quad X=\begin{bmatrix}2\\-2\\1\end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 15. Every

non-zero multiple of $\begin{bmatrix}2\\-2\\1\end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 15

Example 4. Determine the eigen values and eigen vectors of the matrix :

$$A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$$

(H.P.U. 2004, 2008, 2010; P.U. 2005, 2006, 2008; G.N.D.U. 2009, 2010; Pbi. U. 2013)

Sol.

$$A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$$

$$\lambda I=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}$$

$$\therefore \quad A-\lambda I=\begin{bmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{bmatrix}$$

$\therefore$ the characteristic equation of A is $|A-\lambda I|=0$

or $$\begin{vmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{vmatrix}=0$$

or $$\begin{vmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & 2-\lambda\\ 2 & -1 & 2-\lambda\end{vmatrix}=0, \text{ by } C_3 \to C_3+C_2$$

or $$(2-\lambda)\begin{vmatrix}6-\lambda & -2 & 0\\ -2 & 3-\lambda & 1\\ \ldots\end{vmatrix}=0$$