Art-17. Cayley-Hamilton Theorem

(P.U. 2004, 2006, 2012; Pbi. U. 2005, 2007, 2008, 2009, 2010, 2011, 2013; H.P.U. 2004, 2005, 2007, 2010, 2013 ; G.N.D.U. 2006, 2007, 2009, 2010, 2011)

Statement. Every square matrix satisfies its characteristic equation.

Proof. Let A be any square matrix of order n, and its characteristic equation be

$$p_0 + p_1 \lambda + p_2 \lambda^2 + \ldots + p_n \lambda^n = 0$$

We have to prove that A satisfies this equation

i.e., $p_0 I + p_1 A + p_2 A^2 + \ldots + p_n A^n = O$ ...(1)

For proving this, we proceed as follow :

We know that $(A - \lambda I)$ adj. $(A - \lambda I) = |A - \lambda I| I$ $[\because A \text{ adj. } A = |A| I]$

Let adj. $(A - \lambda I) = B_0 + B_1 \lambda + B_2 \lambda^2 + \ldots + B_{n-1} \lambda^{n-1}$

$\therefore$ we have, $(A - \lambda I)(B_0 + B_1 \lambda + B_2 \lambda^2 + \ldots + B_{n-1} \lambda^{n-1})$

$$= (p_0 + p_1 \lambda + p_2 \lambda^2 + \ldots + p_n \lambda^n) I$$

Equating the coefficients of like powers of λ, we get,

$$AB_0 = p_0 I$$

$$AB_1 - B_0 = p_1 I$$

$$AB_2 - B_1 = p_2 I$$

... ...

$$AB_{n-1} - B_{n-2} = p_{n-1} I$$

$$- B_{n-1} = p_n I$$

Pre-multiplying above equations by I, A, A^2,...., A^n respectively and adding , we get,

$O = p_0 I + p_1 A + p_2 A^2 + ... + p_n A^n$, which is same as (1).

Hence the theorem.

Art-18. **Explain how inverse of a square matrix can be found by using Cayley-Hamilton theorem.**

Proof : According to Cayley Hamilton theorem, square matrix A satisfies its characteristic equation.

$$\therefore \quad p_0 I + p_1 A + p_2 A^2 + ... + p_n A^n = O$$

Pre-multiplying both sides by A^{-1}, we get,

$$p_0 A^{-1} I + p_1 A^{-1} A + p_2 A^{-1} A^2 + ... + p_n A^{-1} A^n = O$$

$$\therefore \quad p_0 A^{-1} + p_1 I + p_2 A + ... + p_n A^{n-1} = O$$

$$\therefore \quad p_0 A^{-1} = -(p_1 I + p_2 A + ... + p_n A^{n-1})$$

$$\therefore \quad A^{-1} = -\frac{1}{p_0}(p_1 I + p_2 A + ... + p_n A^{n-1})$$

ILLUSTRATIVE EXAMPLES

Example 1. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ and hence find A^{-1}. (H.P.U. 2013)

Sol. $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\therefore \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2$$

$$= 4 + \lambda^2 - 5\lambda + 2 = \lambda^2 - 5\lambda + 6$$

$\therefore$ the characteristic equation of A is $|A - \lambda I| = 0$ or $\lambda^2 - 5\lambda + 6 = 0$

We have to prove that A satisfies this equation *i.e.* $A^2 - 5A + 6I = 0$ ...(1)

$$A^2 = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1-2 & -2-8 \\ 1+4 & -2+16 \end{bmatrix} = \begin{bmatrix} -1 & -10 \\ 5 & 14 \end{bmatrix}$$

Consider $A^2 - 5A + 6I = \begin{bmatrix} -1 & -10 \\ 5 & 14 \end{bmatrix} - 5\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -10 \\ 5 & 14 \end{bmatrix} + \begin{bmatrix} -5 & 10 \\ -5 & -20 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1-5+6 & -10+10+0 \\ 5-5+0 & 14-20+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore\ A^2 - 5A + 6I = O$

$\therefore$ (1) is satisfied.

Hence the result.

From (1), $A^{-1}(A^2 - 5A + 6\,I) = A^{-1} \cdot O$

$\therefore\ A - 5I + 6\,A^{-1} = O \Rightarrow 6\,A^{-1} = -A + 5I$

$$\therefore\quad 6\,A^{-1} = -\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -1+5 & 2+0 \\ -1+0 & -4+5 \end{bmatrix}$$

$$\therefore\quad 6\,A^{-1} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\therefore\quad A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

Example 2. Find the characteristic equation of the matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$ and verify Cayley Hamilton Theorem. Find A^{-1}.

(Pbi. U. 2013; G.N.D.U. 2010)

Sol.

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore\ |A-\lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 4 & 3-\lambda & 2 \\ 2 & 4 & 3-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 9-\lambda & 9-\lambda & 1 \\ 4 & 3-\lambda & 2 \\ 2 & 4 & 3-\lambda \end{vmatrix}, \text{ by } R_1 \to R_1 + R_2 + R_3$$

$$= (9-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3-\lambda & 2 \\ 2 & 4 & 3-\lambda \end{vmatrix}$$

$$= (9-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 4 & -1-\lambda & -2 \\ 2 & 2 & 1-\lambda \end{vmatrix}, \text{ by } C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$$

$$= (9-\lambda) \begin{vmatrix} -1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (9-\lambda)\,[-1+\lambda^2+4]$$

$$= (9-\lambda)(\lambda^2+3) = -\lambda^3 + 9\lambda^2 - 3\lambda + 27$$

$\therefore$ the characteristic equation of A is $|A-\lambda I| = 0$

or $-\lambda^3 + 9\lambda^2 - 3\lambda + 27 = 0$ or $\lambda^3 - 9\lambda^2 + 3\lambda - 27 = 0$

We have to prove that A satisfies this equation

i.e., $A^3 - 9A^2 + 3A - 27I = O$

$$\text{Now } A^2 = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+8+8 & 6+6+16 & 12+4+12 \\ 12+12+4 & 8+9+8 & 16+6+6 \\ 6+16+6 & 4+12+12 & 8+8+9 \end{bmatrix} = \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 75+112+56 & 50+84+112 & 100+56+84 \\ 84+100+56 & 56+75+112 & 112+50+84 \\ 84+112+50 & 56+84+110 & 112+56+75 \end{bmatrix} = \begin{bmatrix} 243 & 246 & 240 \\ 240 & 243 & 246 \\ 246 & 240 & 243 \end{bmatrix}$$

Consider $A^3 - 9A^2 + 3A - 27I$

$$= \begin{bmatrix} 243 & 246 & 240 \\ 240 & 243 & 246 \\ 246 & 240 & 243 \end{bmatrix} - 9 \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix} + 3 \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} - 27 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 243 & 246 & 240 \\ 240 & 243 & 246 \\ 246 & 240 & 243 \end{bmatrix} - \begin{bmatrix} 225 & 252 & 252 \\ 252 & 225 & 252 \\ 252 & 252 & 225 \end{bmatrix} + \begin{bmatrix} 9 & 6 & 12 \\ 12 & 9 & 6 \\ 6 & 12 & 9 \end{bmatrix} - \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 243-225+9-27 & 246-252+6-0 & 240-252+12-0 \\ 240-252+12-0 & 243-225+9-27 & 246-252+6-0 \\ 246-252+6-0 & 240-252+12-0 & 243-225+9-27 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ result (1) is satisfied.

Hence the result.

From (1), $A^{-1}\ (A^3 - 9A^2 + 3A - 27\ I) = O$

$$\Rightarrow \quad A^2 - 9A + 3I - 27\ A^{-1} = O$$

$$\Rightarrow \quad 27\ A^{-1} = A^2 - 9\ A + 3\ I$$

$$\Rightarrow \quad 27\ A^{-1} = \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix} - 9 \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix} + \begin{bmatrix} -27 & -18 & -36 \\ -36 & -27 & -18 \\ -18 & -36 & -27 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25-27+3 & 28-18+0 & 28-36+0 \\ 28-36+0 & 25-27+3 & 28-18+0 \\ 28-18+0 & 28-36+0 & 25-27+3 \end{bmatrix}$$

$$\therefore \quad 27\ A^{-1} = \begin{bmatrix} 1 & 10 & -8 \\ -8 & 1 & 10 \\ 10 & -8 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{27} \begin{bmatrix} 1 & 10 & -8 \\ -8 & 1 & 10 \\ 10 & -8 & 1 \end{bmatrix}$$

$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$