Art-19. If $m_1(x)$ and $m_2(x)$ be two polynomials of the lowest degree, with scalar coefficients, such that $m_1(A) = O$, $m_2(A) = O$ then each of $m_1(x)$, $m_2(x)$ is a scalar multiple of the other.

Proof : Divide $m_2(x)$ by $m_1(x)$. Let q be the quotient and $r(x)$ be the remainder.

$$\therefore \quad m_2(x) = q\, m_1(x) + r(x) \quad \text{...(1)}$$

where either $r(x) = 0$ or degree of $r(x)$ is less than the degree of $m_1(x)$.

Assume that $r(x) \neq 0$.

From (1), $m_2(A) = q\, m_1(A) + r(A)$

$\Rightarrow \quad O = O + r(A)$

$\Rightarrow \quad r(A) = O$

so that A satisfies an equation of degree lower than that of $m_1(x)$. Thus, we arrive at a contradiction.

$\therefore \quad r(x) = 0$

$\therefore$ from (1), $m_2(x) = q\, m_1(x)$.

Hence the result.

Art-20. Minimal Polynomial and Minimum Equation

Hence the result.

Art-20. Minimal Polynomial and Minimum Equation

If $m(x)$ be a scalar polynomial of the lowest degree with leading coefficient unity, such that $m(x) = 0$ is satisfied by A *i.e.* $m(A) = O$, then the polynomial $m(x)$ is called the minimal polynomial of A and $m(x) = 0$ is called the minimum equation of A.

Note. The degree of the minimal equation of an n-rowed matrix is less than or equal to that of its characteristic equation which is n.

Derogatory and Non-derogatory Matrices

An n-rowed matrix is said to be derogatory or non-derogatory, according as the degree of its minimal equation is less than or equal to n.

Art-21. If $h(x)$ be any polynomial with scalar coefficients, such that $h(A) = O$ and if $m(x) = 0$ be the minimal equal of A, then there exists a polynomial $q(x)$ such that

$$h(x) = m(x)\, q(x)$$

Proof : Divide $h(x)$ by $m(x)$. Let $q(x)$ be the quotient and $r(x)$ be the remainder.

$\therefore$ $$h(x) = q(x)\, m(x) + r(x) \quad ...(1)$$

where either $r(x) = 0$ or degree of $r(x)$ is less than the degree of $m(x)$.

Assume that $r(x) \neq 0$.

From (1) $\quad h(A) = q(A)\, m(A) + r(A)$

$$O = O + r(A)$$

$\therefore$

$\Rightarrow$ $$r(A) = O$$

so that A satisfies an equation of degree lower than that of the minimal equation. Thus we arrive at a contradiction.

$\therefore$ $$r(x) = 0$$

$\therefore$ $$h(x) = m(x)\, q(x).$$

Cor. Minimal polynomial is unique.

If possible, suppose that $m_1(x)$, m_2 are two minimal polynomials of A.

$\therefore$ $m_1(x)$ divides $m_2(x)$ and $m_2(x)$ divides $m_1(x)$. Since $m_1(x)$ and $m_2(x)$ have their leading coefficient unity.

$\therefore$ $$m_1(x) = m_2(x)$$

$\therefore$ minimal polynomial is unique.

Art-22. Prove that

(*i*) each root of minimal equation of A is also a root of characteristic equation of A.

(*ii*) the distinct roots of the characteristic equation of A are also the distinct roots of the minimal equation of A.

(G.N.D.U. 2011)

Proof : (*i*) Let $\phi(x) = 0$ be the characteristic equation and $m(x) = 0$ be the minimal equation of A. Then

$$\phi(x) = q(x)\, m(x).$$

$\Rightarrow$ every root of $m(x) = 0$ is also root of $\phi(x) = 0$.

(*ii*) Let $\phi(x) = 0$ be the characteristic equation

and $m(x) = 0$ be the minimal equation of A.

$\therefore$ there exists a matrix polynomial $L(x)$ such that

$$m(x)\, I = (A - x\, I)\, L(x)$$

$\therefore$ $$|\, m(x)\, I\,| = |\, A - x\, I\,|\, |\, L(x)\,|$$

$\Rightarrow$ $$\{m(x)\}^n = \phi(x)\, |\, L(x)\,|$$

$\Rightarrow$ each root of $\phi(x) = 0$ is also a root of $\{m(x)\}^n = 0$ and thus also of $m(x) = 0$.

Note 1. Counting each repeated root of characteristic equation $\phi(x) = 0$ only once, the set of the roots of $\phi(x) = 0$ is the same as that of $m(x) = 0$. The roots of these two polynomials can only differ in respect of their multiplicities.

Note 2. If the roots of the characteristic equation of an n-rowed matrix A are all different, then its minimal equation is also of nth degree and, in fact, apart from the constant factor $(-1)^n$, it coincides with the characteristic equation and, as such, the matrix is non-derogatory.

Example 1. Find the characteristic equation and the minimal equation of the

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

Also show that A is non-derogatory.

Sol.
$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

or $(8-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} - (-6)\begin{vmatrix} -6 & -4 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -6 & 7-\lambda \\ 2 & -4 \end{vmatrix} = 0$

or $(8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$

or $(8-\lambda)[\lambda^2 - 10\lambda + 21 - 16] + 6(-18 + 6\lambda + 8) + 2(24 - 14 + 2\lambda) = 0$

or $(8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(2\lambda + 10) = 0$

or $-\lambda^3 + 18\lambda^2 - 85\lambda + 40 + 36\lambda - 60 + 4\lambda + 20 = 0$

or $-\lambda^3 + 18\lambda^2 - 45\lambda = 0$ or $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

or $\lambda(\lambda^2 - 18\lambda + 45) = 0$ or $\lambda(\lambda - 3)(\lambda - 15) = 0$

$\therefore$ $\lambda = 0, 3, 15$ are the characteristic roots of A.

Since the characteristic roots of A are all different.

$\therefore$ minimal equation of A is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

Since degree of characteristic equation of A and minimal equation of A is same

$\therefore$ A is non derogatory.

Example 2. Find the minimal polynomial of the matrix :

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -3 \\ 0 & 0 & -2 \end{bmatrix}$$

(G.N.D.U. 2007; H.P.U. 2011)

Sol. The given equation is

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -3 \\ 0 & 0 & -2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 1-\lambda & -2 & 3 \\ 0 & 5-\lambda & -3 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

or $\begin{vmatrix} 1-\lambda & -2 & 3 \\ 0 & 5-\lambda & -3 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

or $(1-\lambda)(5-\lambda)(-2-\lambda) = 0$ [Product of the diagonal elements]

or $(\lambda - 1)(\lambda - 5)(\lambda + 2) = 0$

or $(\lambda^2 - 6\lambda + 5)(\lambda + 2) = 0$

or $\lambda^3 - 4\lambda^2 - 7\lambda + 10 = 0$

Its roots are 1, 5, – 2

Since the characteristic roots of A are all different.

$\therefore$ minimal equation of A is $\lambda^3 - 4\lambda^2 - 7\lambda + 10 = 0$.

Example 3. If $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$, find the minimal polynomial of A. Find A^{-1}, using the minimal polynomial.

(G.N.D.U. 2006)

Sol. $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\therefore \quad \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

or $\quad (5-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ -6 & -4-\lambda \end{vmatrix} - (-6)\begin{vmatrix} -1 & 2 \\ 3 & -4-\lambda \end{vmatrix} - 6\begin{vmatrix} -1 & -4-\lambda \\ 3 & -6 \end{vmatrix} = 0$

or $\quad (5-\lambda)[(4-\lambda)(-4-\lambda)+12] + 6[(-1)(-4-\lambda)-6] - 6[6-3(4-\lambda)] = 0$

or $\quad (5-\lambda)(\lambda^2 - 16 + 12) + 6(4+\lambda-6) - 6(6-12+3\lambda) = 0$

or $\quad -\lambda^3 + 4\lambda + 5\lambda^2 - 20 + 6\lambda - 12 - 18\lambda + 36 = 0$

or $\quad -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

or $\quad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

Putting $\lambda = 1$ in (1), we get

$1 - 5 + 8 - 4 = 0 \quad$ or $\quad 0 = 0$

$\therefore \quad \lambda = 1$ is a roof of (1).

1	1	−5	8	−4
		1	−4	4
	1	−4	4	0

$\therefore$ remaining roots of (1) are given by

$\lambda^2 - 4\lambda + 4 = 0$

$\therefore \quad (\lambda - 2)^2 = 0 \quad$ or $\quad \lambda = 2, 2$

$\therefore \quad \lambda = 1, 2, 2$ are roots of characteristic equation of A.

Distinct roots of characteristic equation of A are 1, 2.

$\therefore$ 1, 2 are also roots of minimal equation of A.

$\therefore$ minimal polynomial of A is either

$(\lambda - 1)(\lambda - 2)$ *i.e.* $\lambda^2 - 3\lambda + 2$ or $\lambda^3 - 5\lambda^2 + 8\lambda - 4$

Now $\quad A^2 - 3A + 2I$

$$=\begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix} + \begin{bmatrix} -15 & 18 & 18 \\ 3 & -12 & -6 \\ -9 & 18 & 12 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} -13+15+2 & -18+18+0 & -18+18+0 \\ -3+3+0 & 10-12+2 & 6-6+0 \\ 9-9+0 & -18+18+0 & -14+12+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $A^2 - 3A + 2I = O$

$\therefore$ minimal equation of A is $\lambda^2 - 3\lambda + 2 = 0$

Now $A^2 - 3A + 2I = O$

Pre-multiplying both sides by A^{-1}, we get,

$$A^{-1}A^2 - 3A^{-1}A + 2A^{-1}I = O$$

$\therefore$ $A - 3I + 2A^{-1} = O$

$\Rightarrow$ $2A^{-1} = -A + 3I$

$$\Rightarrow \quad 2A^{-1} = -\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2A^{-1} = \begin{bmatrix} -5 & 6 & 6 \\ 1 & -4 & -2 \\ -3 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \quad 2A^{-1} = \begin{bmatrix} -5+3 & 6+0 & 6+0 \\ 1+0 & -4+3 & -2+0 \\ -3+0 & 6+0 & 4+3 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 6 & 6 \\ 1 & -1 & -2 \\ -3 & 6 & 7 \end{bmatrix}$$

EXERCISE 6 (*e*)