Art-4. Transform a given equation into another whose roots are the roots of the given equation, each diminished by a number h.

Proof : Let the given equation of nth degree be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be its roots.

We want to form an equation whose roots are $\alpha_1 - h, \alpha_2 - h, \ldots, \alpha_n - h$.

Let new equation be in y

$\therefore \quad y = x - h$

or $\quad x = y + h$

Putting $x = y + h$ in (1), we get the required equation as

or $\quad a_0 (y + h)^n + a_1 (y + h)^{n-1} + a_2 (y + h)^{n-2} + \ldots + a_{n-1} (y + h) + a_n = 0$

which after expanding with the help of Binomial theorem for positive integers takes the form

$$A_0 y^n + A_1 y^{n-1} + A_2 y^{n-2} + \ldots + A_{n-1} y + A_n = 0 \qquad \ldots(2)$$

where $A_0, A_1, A_2, \ldots, A_n$ are to be determined.

Put $y = x - h$ in (2)

$\therefore \quad A_0 (x-h)^n + A_1 (x-h)^{n-1} + A_2 (x-h)^{n-2} + \ldots + A_{n-1} (x-h) + A_n = 0$

or $\quad (x-h) [A_0 (x-h)^{n-1} + A_1 (x-h)^{n-2} + A_2 (x-h)^{n-3} + \ldots$

$$+ A_{n-2} (x-h) + A_{n-1}] + A_n = 0 \qquad \ldots(3)$$

Now L.H.S. of (1) is identical with L.H.S. of (3)

$\therefore$ if we divide $f(x)$ by $x-h$, then A_n is the remainder. Then we divide the quotient

$$A_0 (x-h)^{n-1} + A_1 (x-h)^{n-2} + \ldots + A_{n-2} (x-h) + A_{n-1}$$

by $x - h$, we get A_{n-1} as the remainder

Proceeding in this way, we get $A_{n-2}, \ldots, A_0$.

Method. (*i*) Make the equation $f(x) = 0$ complete, if it is not, by supplying zeros as coefficients of missing terms.

(*ii*) Divide $f(x)$ by $x - h$ and denote the remainder by A_n.

(*iii*) Divide the quotient obtained in step (*ii*) by $x - h$ to obtain remainder A_{n-1}.

(*iv*) Continue with this process upto A_0.

Note. In case each root is to be increased by h, then the division should be done by $x + h$.

Art-5. Reduce the cubic $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ to the form in which the second term is missing and coefficient of the leading term is unity, all other coefficients being integers.

(Pbi. U. 2005, 2009, 2011; H.P.U. 2008, 2009)

Proof : The given equation is

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \quad \ldots(1)$$

Let us diminish each root of it by h. Let new equation be in y.

$\therefore \quad y = x - h \quad$ or $\quad x = y + h$

Putting $x = y + h$ in (1), we get,

$$a_0(y+h)^3 + 3a_1(y+h)^2 + 3a_2(y+h) + a_3 = 0$$

or $\quad a_0(y^3 + 3y^2h + 3yh^2 + h^3) + 3a_1(y^2 + 2yh + h^2) + 3a_2(y+h) + a_3 = 0$

or $\quad a_0y^3 + (a_0h + a_1)y^2 + 3(a_0h^2 + 2a_1h + a_2)y$

$$+ (a_0h^3 + 3a_1h^2 + 3a_2h + a_3) = 0 \quad \ldots(2)$$

Second term will be missing when coefficient of $y^2 = 0$

$$\therefore \quad a_0h + a_1 = 0 \quad \Rightarrow \quad h = -\frac{a_1}{a_0}$$

Now in equation (2), coefficient of

$$y = 3(a_0h^2 + 2a_1h + a_2)$$

$$= 3\left[a_0\left(\frac{a_1^2}{a_0^2}\right) + 2a_1\left(-\frac{a_1}{a_0}\right) + a_2\right] = 3\left[\frac{a_1^2}{a_0} - \frac{2a_1^2}{a_0} + a_2\right]$$

$$= 3\left[\frac{a_1^2 - 2a_1^2 + a_0a_2}{a_0}\right] = 3\frac{(a_0a_2 - a_1^2)}{a_0} = \frac{3H}{a_0} \quad \text{where } H = a_0a_2 - a_1^2$$

Constant term $= a_0h^3 + 3a_1h^2 + 3a_2h + a_3$

$$= a_0\left(-\frac{a_1^3}{a_0^3}\right) + 3a_1\left(\frac{a_1^2}{a_0^2}\right) + 3a_2\left(-\frac{a_1}{a_0}\right) + a_3$$

$$= -\frac{a_1^3}{a_0^2} + \frac{3a_1^3}{a_0^2} - \frac{3a_1a_2}{a_0} + a_3 = \frac{-a_1^3 + 3a_1^3 - 3a_0a_1a_2 + a_0^2a_3}{a_0^2}$$

$$= \frac{2a_1^3 - 3a_0a_1a_2 + a_0^2a_3}{a_0^2}$$

$$= \frac{G}{a_0^2} \quad \text{where } G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$$

∴ equation (2) becomes

$$a_0 y^3 + \frac{3H}{a_0} y + \frac{G}{a_0^2} = 0$$

or
$$y^3 + \frac{3H}{a_0^2} y + \frac{G}{a_0^3} = 0$$

Multiply its each roots by a_0. Let new equation be in z.

$$\therefore \quad a_0^0 z^3 + a_0^2 \cdot \frac{3H}{a_0^2} z + a_0^3 \frac{G}{a_0^3} = 0$$

or
$$z^3 + 3Hz + G = 0$$

which is required equation.

Cor. The relation between roots of the equation

$z^3 + 3Hz + G = 0$ and that of equation (1) is

$$z = a_0 y = a_0(x-h) = a_0\left(x + \frac{a_1}{a_0}\right)$$

Let x_1, x_2, x_3 be roots of equation (1) and z_1, z_2, z_3 be corresponding roots of equation

$$z^3 + 3Hz + G = 0$$

$$\therefore \quad x_1 + x_2 + x_3 = -3\frac{a_1}{a_0}$$

$$\therefore \quad \frac{a_1}{a_0} = -\frac{1}{3}(x_1 + x_2 + x_3)$$

$$\therefore \quad z_1 = a_0\left[x_1 - \frac{1}{3}(x_1 + x_2 + x_3)\right]$$

$$\therefore \quad z_1 = \frac{a_0}{3}(2x_1 - x_2 - x_3)$$

Similarly $z_2 = \dfrac{a_0}{3}(2x_2 - x_3 - x_1)$

$$z_3 = \frac{a_0}{3}(2x_3 - x_1 - x_2)$$

Art-6. Reduce the quadratic $a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = 0$ to the form in which the second term is missing and the coefficient of the leading term is unity, all other coefficients being integers. (G.N.D.U. 2012)

Proof : The given equation is

$$a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 + a_4 = 0 \qquad \ldots(1)$$

Let us diminish each root of it by h.

Let new equation be in y.

$\therefore \quad y = x - h, \quad \text{or} \quad x = y + h$

Putting $x = y + h$ in (1), we get,

$$a_0 (y + h)^4 + 4 a_1 (y + h)^3 + 6 a_2 (y + h)^2 + 4 a_3 (y + h) + a_4 = 0$$

$$\therefore \quad a_0 (y^4 + 4 y^3 h + 6 y^2 h^2 + 4 y h^3 + h^4) + 4 a_1 (y^3 + 3 y^2 h + 3 y h^2 + h^3)$$
$$+ 6 a_2 (y^2 + 2 y h + h^2) + 4 a_3 (y + h) + a_4 = 0$$

or $$a_0 y^4 + 4 (a_0 h + a_1) y^3 + 6 (a_0 h^2 + 2 a_1 h + a_2) y^2$$
$$+ 4 (a_0 h^3 + 3 a_1 h^2 + 3 a_2 h + a_3) y$$
$$+ (a_0 h^4 + 4 a_1 h^3 + 6 a_2 h^2 + 4 a_3 h + a_4) = 0 \quad \ldots(2)$$

Second term will be missing when $a_0 h + a_1 + 0$ *i.e.*, $h = -\dfrac{a_1}{a_0}$

$\therefore$ coefficients of $y^2 = 6 (a_0 h^2 + 2 a_1 h + a_2)$

$$= 6 \left[a_0 \frac{a_1^2}{a_0^2} - 2 a_1 \cdot \frac{a_1}{a_0} + a_2 \right] = 6 \left[\frac{a_1^2}{a_0} - \frac{2 a_1^2}{a_0} + a_2 \right] = 6 \left[\frac{- a_1^2 + a_0 a_2}{a_0} \right]$$

$$= \frac{6H}{a_0} \quad \text{where } H = a_0 a_2 - a_1^2$$

Coefficient of $y = 4 [a_0 h^3 + 3 a_1 h^2 + 3 a_2 h + a_3]$

$$= 4 \left[- a_0 \cdot \frac{a_1^3}{a_0^3} + 3 a_1 \frac{a_1^2}{a_0^2} - 3 a_2 \cdot \frac{a_1}{a_0} + a_3 \right]$$

$$= 4 \left[- \frac{a_1^3}{a_0^2} + \frac{3 a_1^3}{a_0^2} - \frac{3 a_1 a_2}{a_0} + a_3 \right] = 4 \left[\frac{2 a_1^3}{a_0^2} - \frac{3 a_1 a_2}{a_0} + a_3 \right]$$

$$= 4 \left[\frac{2 a_1^3 - 3 a_0 a_1 a_2 + a_0^2 a_3}{a_0^2} \right] = 4 \frac{G}{a_0^2}, \text{ where G}$$

$$= a_0^2 a_3 - 3 a_0 a_1 a_2 + 2 a_1^3$$

Constant term $= a_0 h^4 + 4 a_1 h^3 + 6 a_2 h^2 + 4 a_3 h + a_4$

$$= a_0 \frac{a_1^4}{a_0^4} - 4 a_1 \cdot \frac{a_1^3}{a_0^3} + 6 a_2 \frac{a_1^2}{a_0^2} - 4 a_3 \frac{a_1}{a_0} + a_4$$

$$= \frac{a_1^4}{a_0^3} - \frac{4a_1^4}{a_0^3} + \frac{6a_1^2 a_2}{a_0^2} - \frac{4a_1 a_3}{a_0} + a_4$$

$$= \frac{a_1^4 - 4a_1^4 + 6a_0 a_1^2 a_2 - 4a_0^2 a_1 a_3 + a_0^3 a_4}{a_0^3}$$

$$= \frac{-3a_1^4 + 6a_0 a_1^2 a_2 - 4a_0^2 a_1 a_3 + a_0^2 a_4}{a_0^3}$$

$$= \frac{a_0^2 (a_0 a_4 - 4a_1 a_3 + 3a_2^2) - 3(a_0 a_2 - a_1^2)^2}{a_0^3}$$

$$= \frac{a_0^2 \mathrm{I} - 3\mathrm{H}^2}{a_0^2}, \text{ where } \mathrm{I} = a_0 a_4 - 4a_1 a_3 + 3a_2^2$$

$\therefore$ equation (2) becomes

$$a_0 y^4 + \frac{6\mathrm{H}}{a_0} y^2 + \frac{4\mathrm{G}}{a_0^2} y + \frac{a_0^2 \mathrm{I} - 3\mathrm{H}^2}{a_0^3} = 0$$

or $$y^4 + \frac{6\mathrm{H}}{a_0^2} y^2 + \frac{4\mathrm{G}}{a_0^3} y + \frac{a_0^2 \mathrm{I} - 3\mathrm{H}^2}{a_0^4} = 0$$

Multiplying its roots by a_0, we get,

$$z^4 + 6\mathrm{H} z^2 + 4\mathrm{G} z - (a_0^2 \mathrm{I} - 3\mathrm{H}^2) = 0$$

which is the required equation.

Cor. Let z_1, z_2, z_3, z_4 be roots of transformed equation

$$z^4 + 6\mathrm{H} z^2 + 4\mathrm{G} z + (a_0^2 \mathrm{I} - 3\mathrm{H}^2) = 0$$

and x_1, x_2, x_3, x_4 be corresponding roots of (1)

$$\therefore \quad z_1 = a_0 \left(x_1 + \frac{a_1}{a_0} \right)$$

But $x_1 + x_2 + x_3 + x_4 = -4 \dfrac{a_1}{a_0}$

$$\therefore \quad \frac{a_1}{a_0} = -\frac{1}{4} (x_1 + x_2 + x_3 + x_4)$$

$$\therefore \quad z_1 = a_0 \left[x_1 - \frac{1}{4} (x_1 + x_2 + x_3 + x_4) \right]$$

$$z_1 = \frac{a_0}{4} (3x_1 - x_2 - x_3 - x_4)$$

Similarly $z_2 = \frac{a_0}{4}(3x_2 - x_3 - x_4 - x_1)$

$$z_3 = \frac{a_0}{4}(3x_3 - x_4 - x_1 - x_2)$$

$$z_4 = \frac{a_0}{4}(3x_4 - x_1 - x_2 - x_3)$$

ILLUSTRATIVE EXAMPLES

Example 1. Diminish the roots of the equation $2x^5 - x^3 + 10x - 8 = 0$ by 5.

Sol. The given equation is $2x^5 - x^3 + 10x - 8 = 0$

or $2x^5 + 0x^4 - x^3 - 0x^2 + 10x - 8 = 0$ ...(1)

We are to diminish each root by 5

5	2	0	− 1	0	10	− 8
		10	50	245	1225	7175
	2	10	49	245	1235	$6167 = A_5$
		10	100	745	4950	
	2	20	149	990	$6185 = A_4$	
		10	150	1495		
	2	30	299	$2485 = A_3$		
		10	200			
	2	40	$499 = A_2$			
		10				
	2	$50 = A_1$				
	$2 = A_0$					

∴ transformed equation is

$$2y^5 + 50y^4 + 499y^3 + 2485y^2 + 6185y + 6167 = 0.$$

Example 2. Increase by 4 the roots of the equation $3x^5 - 5x^3 + 7 = 0$.

(Pbi. U. 2008, 2010)

Sol. The given equation is $3x^5 - 5x^3 + 7 = 0$

or $3x^5 + 0x^4 - 5x^3 + 0x^2 + 0x + 7 = 0$

We are to increase each root by 4.

Let new equation be in y

-4	3	0	-5	0	0	7
		-12	48	-172	688	-2752
	3	-12	43	-172	688	-2745
		-12	96	-556	2912	
	3	-24	139	-728	3600	
		-12	144	-1132		
	3	-36	283	-1860		
		-12	192			
	3	-48	475			
		-12				
	3	-60				
	3					

∴ transformed equation is

$$3y^5 - 60y^4 + 475y^3 - 1860y^2 + 3600y - 2745 = 0.$$

Example 3. Find the equation whose roots exceed by 2 the roots of the equation $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$. Hence find the roots of the equation.

(H.P.U. 2002)

Sol. The given equation is

$$4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0 \quad \text{...(1)}$$

-2	4	32	83	76	21
		-8	-48	-70	-12
	4	24	35	6	9
		-8	-32	-6	
	4	16	3	0	
		-8	-16		
	4	8	-13		
		-8			
	4	0			
	4				

∴ transformed equation is $4y^4 - 13y^2 + 9 = 0$

∴ $4y^4 - 4y^2 - 9y^2 + 9 = 0$

∴ $4y^2(y^2 - 1) - 9(y^2 - 1) = 0$

∴ $(y^2 - 1)(4y^2 - 9) = 0$

∴ $y^2 = 1, \frac{9}{4}$

$\therefore \quad y = 1, -1, \frac{3}{2}, -\frac{3}{2}$

Now $\quad y = x + 2$

$\therefore \quad x = y - 2$

$\therefore$ roots of equation (1) are $1 - 2, -1 - 2, \frac{3}{2} - 2, -\frac{3}{2} - 2$

or $\quad -1, -3, -\frac{1}{2}, -\frac{7}{2}$.

Example 4. Transform the equation $x^3 - 6x^2 + 11x - 6 = 0$ by the transformation $y = x -$ and hence find all roots of the given equation.

(G.N.D.U. 2000; Pbi. U. 20

Sol. The given equation is $\quad x^3 - 6x^2 + 11x - 6 = 0$ ...(

The transformation is $y = x - 2$

It means that the new equation is in y and we are to diminish each root of (1) by 2.

2	1	−6	11	−6
		2	−8	6
	1	−4	3	0
		2	−4	
	1	−2	−1	
		2		
	1	0		
	1			

$\therefore$ transformed equation is

$$y^3 - y = 0$$

$\therefore \quad y(y^2 - 1) = 0$

$\therefore \quad y(y - 1)(y + 1) = 0$

$\therefore \quad y = 0, 1, -1.$

Now $\quad y = x - 2 \quad \Rightarrow \quad x = y + 2$

$\therefore$ roots of equation (1) are $0 + 2, 1 + 2, -1 + 2$ or $2, 3, 1$.

Example 5. Remove the second term from the equation $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$ and hence solve it completely.

(G.N.D.U. 2001, 2002, 2005; P.U. 2010; H.P.U. 20

Sol. The given equation is

$$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$$

Sum of four roots = 16

$\therefore$ second term will be missing when we decrease each root by 4.

4	1	−16	86	−176	105
		4	−48	152	−96
	1	−12	38	−24	9
		4	−32	24	
	1	−8	6	0	
		4	−16		
	1	−4	−10		
		4			
	1	0			
	1				

Let new equation be in y

$\therefore$ transformed equation is $y^4 - 10\,y^2 + 9 = 0$

$\therefore$ $(y^2 - 1)(y^2 - 9) = 0$

$\therefore$ $y^2 = 1, 9$

$\therefore$ $y = -1, 1, -3, 3$

Now $y = x - 4$ $\Rightarrow$ $x = y + 4$

$\therefore$ roots of (1) are $-1 + 4, 1 + 4, -3 + 4, 3 + 4$ or $3, 5, 1, 7$

Example 6. Transform the equation $x^4 - 4\,x^3 - 18\,x^2 - 3\,x + 2 = 0$ into one which shall want the third term.

Sol. The given equation is $x^4 - 4\,x^3 - 18\,x^2 - 3\,x + 2 = 0$ ...(1)

Let us diminish each root by h.

Let new equation be in y

h	1	-4	-18	-3	2
		h	$h^2 - 4h$	$h^3 - 4h^2 - 18h$	$h^4 - 4h^3 - 18h^2 - 3h$
	1	$h - 4$	$h^2 - 4h - 18$	$h^3 - 4h^2 - 18h - 3$	$h^4 - 4h^3 - 18h^2 - 3h + 2$
		h	$2h^2 - 4h$	$3h^3 - 8h^2 - 18h$	
	1	$2h - 4$	$3h^2 - 8h - 18$	$4h^3 - 12h^2 - 36h - 3$	
		h	$3h^2 - 4h$		
	1	$3h - 4$	$6h^2 - 12h - 18$		
		h			
	1	$4h - 4$			
	1				

∴ transformed equation is

$$y^4 + 4(h-1)y^3 + 6(h^2 - 2h - 3)y^2 + (4h^3 - 12h^2 - 36h - 3)y$$
$$+ (h^4 - 4h^3 - 18h^2 - 3h + 2) = 0 \quad ...(2)$$

Now third term will be missing when coefficient of $y^2 = 0$

∴ $6(h^2 - 2h - 3) = 0 \Rightarrow h^2 - 2h - 3 = 0$

∴ $(h+1)(h-3) = 0 \Rightarrow h = -1, 3$

∴ third term will be missing if we diminish each root of (1) either by -1 or by 3

– 1	1	– 4	– 18	– 3	2
		– 1	5	13	– 10
	1	– 5	– 13	10	– 8
		– 1	6	7	
	1	– 6	– 7	17	
		– 1	7	17	
	1	– 7	0		
		– 1			
	1	– 8			
	1				

∴ transformed equation is $y^4 - 8y^3 + 17y - 8 = 0$

Let us diminish each root by 3

3	1	– 4	– 18	– 3	2
		3	– 3	– 63	– 198
	1	– 1	– 21	– 66	– 196
		3	6	– 45	
	1	2	– 15	– 111	
		3	15		
	1	5	0		
		3			
	1	8			
	1				

∴ transformed equation is $y^4 + 8y^3 - 111y - 196 = 0$.