## Art-8. General Transformations

Method to form an equation whose roots are the symmetric functions of roots of the given equation $f(x) = 0$

*(i)* Write down the relations between the roots and coefficients of the given equation.

*(ii)* Express one of the given roots in terms of only one root.

*(iii)* Change that one root in $x$ and put $y$ = any one of the given roots in $x$.

*(iv)* Eliminate $x$ between this relation and given equation.

**Example 1.** If $\alpha, \beta, \gamma$ are roots of $x^3 - 12x^2 + 39x - 28 = 0$, find the equation whose roots are $\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}, \frac{1}{\gamma - 2}$. (H.P.U. 2006)

**Sol.** The given equation is $x^3 - 12x^2 + 39x - 28 = 0$ ...(1)

Its roots are $\alpha, \beta, \gamma$

We are to form an equation whose roots are $\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}, \frac{1}{\gamma - 2}$.

Let new equation be in $y$

$$\therefore \quad y = \frac{1}{x-2} \quad \text{or} \quad x - 2 = \frac{1}{y}$$

$$\therefore \quad x = 2 + \frac{1}{y} \quad \text{or} \quad x = \frac{2y+1}{y}$$

Putting $x = \frac{2y+1}{y}$ in (1), we get,

$$\left(\frac{2y+1}{y}\right)^3 - 12\left(\frac{2y+1}{y}\right)^2 + 39\left(\frac{2y+1}{y}\right) - 28 = 0$$

or $$\frac{(2y+1)^3}{y^3} - \frac{12(2y+1)^2}{y^2} + 39\frac{(2y+1)}{y} - 28 = 0$$

or $$(2y+1)^3 - 12y(2y+1)^2 + 39y^2(2y+1) - 28y^3 = 0$$

or $$(8y^3 + 12y^2 + 6y + 1) - 12y(4y^2 + 4y + 1) + 39y^2(2y+1) - 28y^3 = 0$$

or $$8y^3 + 12y^2 + 6y + 1 - 48y^3 - 48y^2 - 12y + 78y^3 + 39y^2 - 28y^3 = 0$$

or $$10y^3 + 3y^2 - 6y + 1 = 0$$

which is required equation.

**Example 2.** If $\alpha, \beta, \gamma$ are the roots of $x^3 + ax^2 + bx + c = 0$, form an equation whose root are $\frac{\alpha}{\beta + \gamma}, \frac{\beta}{\gamma + \alpha}, \frac{\gamma}{\alpha + \beta}$.

(H.P.U. 2009; P.U. 2009)

**Sol.** The given equation is $x^3 + ax^2 + bx + c = 0$ ...(1)

$\alpha, \beta, \gamma$ are its roots

$$\therefore \quad \alpha + \beta + \gamma = -a \qquad \ldots(2)$$

Now $\dfrac{\alpha}{\beta+\gamma} = \dfrac{\alpha}{(\alpha+\beta+\gamma)-\alpha} = \dfrac{\alpha}{-a-\alpha}$ [∵ of (2)]

Let new equation be in $y$

$$\therefore \quad y = \frac{x}{-a-x}$$

$$\therefore \quad -ay - xy = x \quad \Rightarrow \quad -ay = xy + x$$

$$\therefore \quad x(y+1) = -ay \quad \Rightarrow \quad x = -\frac{ay}{y+1}$$

Putting this value of $x$ in (1), we get,

$$-\frac{a^3y^3}{(y+1)^3} + a\cdot\frac{a^2y^2}{(y+1)^2} - b\cdot\frac{ay}{y+1} + c = 0$$

or $\quad -a^3y^3 + a^3y^2(y+1) - aby(y+1)^2 + c(y+1)^3 = 0$

or $\quad -a^3y^3 + a^3y^2(y+1) - aby(y^2+2y+1) + c(y^3+3y^2+3y+1) = 0$

or $\quad -a^3y^3 + a^3y^3 + a^3y^2 - aby^3 - 2aby^2 - aby + cy^3 + 3cy^2 + 3cy + c = 0$

or $\quad (c-ab)y^3 + (3c + a^3 - 2ab)y^2 + (3c - ab)y + c = 0$

which is required equation.

**Example 3.** If $\alpha, \beta, \gamma$ are roots of $x^3 - ax^2 + bx - c = 0$, find an equation whose roots are $\beta\gamma - \dfrac{1}{\alpha}, \gamma\alpha - \dfrac{1}{\beta}, \alpha\beta - \dfrac{1}{\gamma}$ (G.N.D.U. 2003)

**Sol.** The given equation is

$$x^3 - ax^2 + bx - c = 0 \qquad \ldots(1)$$

∵ $\alpha, \beta, \gamma$ are its roots

$$\therefore \quad \alpha\beta\gamma = c \qquad \ldots(2)$$

Now $\beta\gamma - \dfrac{1}{\alpha} = \dfrac{\alpha\beta\gamma - 1}{\alpha} = \dfrac{c-1}{\alpha}$ [∵ of (2)]

Let new equation be in $y$

$$\therefore \quad y = \frac{c-1}{x} \quad \text{or} \quad x = \frac{c-1}{y}$$

Putting this value of $x$ in (1), we get,

$$\frac{(c-1)^3}{y^3} - \frac{a(c-1)^2}{y^2} + \frac{b(c-1)}{y} - c = 0$$

or $\quad (c-1)^3 - a(c-1)^2y + b(c-1)y^2 - cy^3 = 0$

or $\quad cy^3 - b(c-1)y^2 + a(c-1)^2y - (c-1)^3 = 0$

which is required equation.