Art-9. Find an equation whose roots are squares of the differences of the roots of the equation $x^3 + 3Hx + G = 0$ $(G \neq 0)$.

(H.P.U. 2001; P.U. 2006)

Proof : The given equation is

$$x^3 + 3Hx + G = 0 \qquad \ldots(1)$$

Let α, β, γ be its roots

$$\left.\begin{aligned} \therefore \quad & \alpha + \beta + \gamma = 0 \\ \text{and} \quad & \alpha\beta\gamma = -G \end{aligned}\right\} \qquad \ldots(2)$$

We want to form an equation whose roots are

$$(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2.$$

Now $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = (-\gamma)^2 - \frac{4\alpha\beta\gamma}{\gamma}$ [∵ of (2)]

$$= \gamma^2 + \frac{4G}{\gamma}$$ [∵ of (2)]

$$= \frac{\gamma^3 + 4G}{\gamma}$$

Let new equation be in y

$\therefore \quad y = \frac{x^3 + 4G}{x}$

$\therefore \quad xy = x^3 + 4G$

$\therefore \quad x^3 - xy + 4G = 0$ ...(3)

Subtracting (3) from (2), we get,

$2Hx + xy - 3G = 0 \Rightarrow (3H + y)x = 3G$

$\therefore \quad x = \frac{3G}{y + 3H}$

Putting this value of x in (1), we get,

$$\frac{27G^3}{(y+3H)^3} + \frac{9GH}{y+3H} + G = 0$$

or $$(y+3H)^3 + 9H(y+3H)^2 + 27G^2 = 0$$

$\therefore \quad y^3 + 9Hy^2 + 27H^2y + 27H^3 + 9Hy^2 + 54H^2y + 81H^3 + 27G^2 = 0$

$\therefore \quad y^3 + 18Hy^2 + 81H^2y + 27(G^2 + 4H^3) = 0$

which is the required equation.

Art-10. Find the equation whose roots are squares of differences of the roots of the cubic $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$.

Proof : The given equation is

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

Let x_1, x_2 and x_3 be its roots.

We have to form an equation whose roots are $(x_1-x_2)^2, (x_2-x_3)^2, (x_3-x_1)^2$.

Now equation (1) reduces to the form

$y^3 + 3Hy + G = 0$ ...(2)

where $y = a_0x + a_1$

$H = a_0a_2 - a_1^2,\ G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$

Let y_1, y_2, y_3 be roots of (2)

$\therefore \quad \left.\begin{array}{l} y_1 + y_2 + y_3 = 0 \\ y_1y_2y_3 = -G \end{array}\right\}$ ...(3)

Also $y_1 = a_0 x_1 + a_1$

$y_2 = a_0 x_2 + a_1$

$y_3 = a_0 x_3 + a_1$

$\therefore \quad y_1 - y_2 = a_0 (x_1 - x_2)$

$y_2 - y_3 = a_0 (x_2 - x_3)$

$y_3 - y_1 = a_0 (x_3 - x_1)$

Let us form an equation whose roots are

$(y_1 - y_2)^2, (y_2 - y_3)^2, (y_3 - y_1)^2$.

Let new equation be in z.

$\therefore$ transformed equation is

$$z^3 + 18\text{H} z^2 + 81\text{H}^2 z + 27(\text{G}^2 + 4\text{H}^3) = 0 \quad \ldots(4)$$

Let required equation be in t

$$\therefore \quad t = (x_2 - x_3)^2 = \frac{(y_2 - y_3)^2}{a_0^2} \Rightarrow t = \frac{z}{a_0^2}$$

$\therefore \quad z = a_0^2 t$

Putting this value of z in (4), we get,

$a_0^6 t^3 + 18\text{H}\, a_0^4 t^2 + 81\text{H}^2 a_0^2 t + 27(\text{G}^2 + 4\text{H}^3) = 0$, which is required equation.

ILLUSTRATIVE EXAMPLES

Example 1. If α, β, γ are roots of $x^3 + 3x + 2 = 0$, form an equation whose roots are $(\beta - \gamma)^2, (\gamma - \alpha)^2$ and $(\alpha - \beta)^2$. (Pbi. U. 2005, 2008, 2011; H.P.U. 2010)

Hence show that the equation has a pair of imaginary roots. (G.N.D.U. 2002)

Sol. The given equation is $x^3 + 3x + 2 = 0$ ...(1)

$\because \alpha, \beta, \gamma$ are its roots

$$\therefore \quad \left.\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \alpha\beta\gamma &= -2 \end{aligned}\right\} \quad \ldots(2)$$

Now $(\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta\gamma = (-\alpha)^2 - \dfrac{4\alpha\beta\gamma}{\alpha}$ [$\because$ of (2)]

$$= \alpha^2 + \frac{8}{\alpha} = \frac{\alpha^3 + 8}{\alpha}$$

Let new equation be in y

$$\therefore \quad y = \frac{x^3 + 8}{x} \Rightarrow xy = x^3 + 8$$

$$x^3 - xy + 8 = 0 \quad \ldots(3)$$

Subtracting (3) from (1), we get,

$$3x + xy - 6 = 0 \qquad \Rightarrow \qquad x(y+3) = 6$$

$$\therefore \qquad x = \frac{6}{x+3}$$

Putting this value of x in (1), we get,

$$\frac{216}{(x+3)^3} + 3\frac{6}{y+3} + 2 = 0$$

$\therefore \quad 216 + 18(y+3)^2 + (y+3)^3 = 0$

$\therefore \quad (y+3)^3 + 9(y+3)^2 + 108 = 0$

$\therefore \quad y^3 + 9y^2 + 27y + 27 + 9y^2 + 54y + 81 + 108 = 0$

$\therefore \quad y^3 + 18y^2 + 81y + 216 = 0$, which is the required equation.

Its roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$

$\therefore \quad (\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2 = -216$

$\therefore$ either one of the three factors $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$ is negative or all the three are negative.

Now all the three factors cannot be negative as a cubic has atleast one real root.

$\therefore$ only one factor $(\alpha - \beta)^2$, say, is negative.

$\therefore$ $\alpha - \beta$ is purely imaginary.

$\Rightarrow$ α and β are conjugate complex numbers.

$\therefore$ given equation has a pair of imaginary roots.

Example 2. Find the equation of the squared differences of the roots of the cubic

$$x^3 - 9x^2 + 23x - 15 = 0. \qquad \text{(H.P.U. 2007)}$$

Sol. The given equation is $x^3 - 9x^2 + 23x - 15 = 0$ ...(I)

Let α, β, γ be its roots.

Sum of three roots = 9

$\therefore$ second term will be removed when we decrease each root by 3.

3	1	−9	23	−15
		3	−18	15
	1	−6	5	0
		3	−9	
	1	−3	−4	
		3		
	1	0		
	1			

Let new equation be in y

$\therefore$ transformed equation is $y^3 - 4y = 0$ or $y(y^2 - 4) = 0$

$\therefore$ $y = 0, -2, 2$

$\therefore$ $\alpha = 0 + 3,\ \beta = -2 + 3,\ \gamma = 2 + 3$

$\Rightarrow$ $\alpha = 3,\ \beta = 1,\ \gamma = 5$ $\Rightarrow$ $\alpha - \beta = 2,\ \beta - \gamma = -4,\ \gamma - \alpha = 2$

$\Rightarrow$ $(\alpha - \beta)^2 = 4,\ (\beta - \gamma)^2 = 16,\ (\gamma - \alpha)^2 = 4$

$\therefore$ we want to form an equation whose roots are 4, 4, 16.

$\therefore$ required solution is $(t - 4)(t - 4)(t - 16) = 0$

or $t^3 - 24t^2 + 144t - 256 = 0.$

EXERCISE 8 (*e*)

1. If α, β, γ are roots of equation $x^3 + 6x + 2 = 0$, then form a cubic equation having roots $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$. (G.N.D.U. 2006)
2. If α, β, γ are the roots of the cubic $x^3 - 3x + 1 = 0$, form an equation whose roots are $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$. (H.P.U. 2011, 2013)
3. Find the equation of the required differences of $x^3 - 7x + 6 = 0$.
4. Find the equation whose roots are the squared differences of the roots of the cubic $x^3 - 6x + 4\sqrt{2} = 0$.

   Hence prove that the given equation has a multiple root.

   (P.U. 2000, 2001, 2007; Pbi. U. 2004)
5. Find an equation whose roots are the squares of the differences of the roots of $x^3 + qx + r = 0$. (G.N.D.U. 2003)
6. Find an equation whose roots are the squares of the differences of the roots of the