3

INTERSECTION OF THREE PLANES

Art-1. Intersection of Three Planes

Case I. Find the condition that three given planes may intersect at a point.

Proof. Let the equations of three planes be

$$a_1x + b_1 y + c_1z + d_1 = 0 \quad \ldots(1)$$

$$a_2x + b_2 y + c_2 z + d_2 = 0 \quad \ldots(2)$$

$$a_3 x + b_3 y + c_3 z + d_3 = 0 \quad \ldots(3)$$

If the three planes intersect at a point, then the line of intersection of any two of the planes, say, (1) and (2) intersects the third plane (3) i.e., the line of intersection of the planes (1) and (2) is not parallel to the plane (3).

First of all, we find the equations of the line of intersection of planes (1) and (2) in the symmetrical form.

Omitting the constant terms from (1) and (2), we get,

$$a_1 x + b_1 y + c_1 z = 0$$

$$a_2 x + b_2 y + c_2 z = 0$$

$$\therefore \quad \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

$\therefore$ *d. cs.* of the line of intersection of planes (1) and (2) are proportional to $b_1 c_2 - b_2 c_1$, $c_1 a_2 - c_2 a_1$, $a_1 b_2 - a_2 b_1$.

Putting $z = 0$ in (1) and (2), we get,

$$a_1 x + b_1 y + d_1 = 0$$

$$a_2 x + b_2 y + d_2 = 0$$

$$\therefore \quad \frac{x}{b_1d_2 - b_2d_1} = \frac{y}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \quad y = \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, \quad z = 0$$

$\therefore$ one point on the line of intersection of planes (1) and (2) is

$$\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right)$$

∴ the equations of the line of intersection of the planes (1) and (2) are

$$\frac{x-\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1}=\frac{y-\dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1}$$

If this line is not parallel to the plane (3), then its is not perpendicular to the normal to the plane (3).

∴ $(b_1 c_2 - b_2 c_1) a_3 + (c_1 a_2 - c_2 a_1) b_3 + (a_1 b_2 - a_2 b_1) c_3 \neq 0$

or $a_3 (b_1 c_2 - b_2 c_1) - b_3 (c_2 a_1 - c_1 a_2) + c_3 (a_1 b_2 - a_2 b_1) \neq 0$

or $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

which is the required condition.

Note 1. The given planes intersect at a point if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

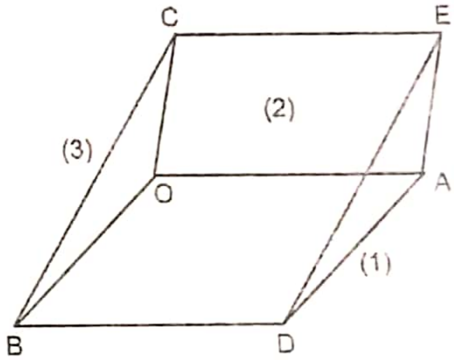
Note 2. The point of intersection is obtained by solving given equations for x, y, z. The use of Crammer's Rule helps us to find the point easily. But equations can be solved by any method.

Note on Triangular Prism

A triangular prism is a figure.

(*i*) whose end faces are two congruent and parallel triangles OBC and ADE, and

(*ii*) side faces are parallelograms OADB, OCEA, BCED.



It should be noted properly that OA, the line of intersection of planes (1) and (2), CE the line of intersection of (2) and (3), and BD the line of intersection of planes (3) and (1) are all parallel.

Case II. Find the condition that the three given planes may form a triangular prism.

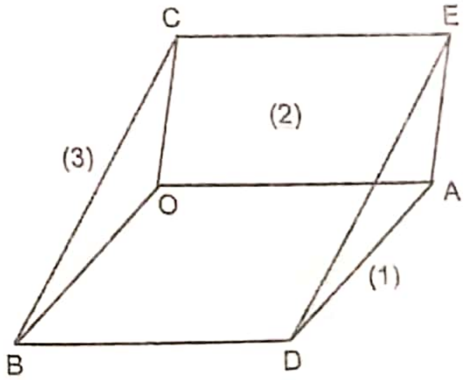
Proof. Let the equations of the planes be

$a_1 x + b_1 y + c_1 z + d_1 = 0$ ...(1)

$a_2 x + b_2 y + c_2 z + d_2 = 0$ ...(2)

$a_3 x + b_3 y + c_3 z + d_3 = 0$ ...(3)

If the three planes form a triangular prism, then the line of intersection of any two of the planes, say, (1) and (2) is parallel to the third plane (3).



First of all we find the equations of the line of intersection of planes (1) and (2) in the symmetrical form.

Omitting the constant terms form (1) and (2), we get,

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$

$$\therefore \quad \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

$\therefore$ direction cosines of the line of intersection of planes (1) and (2) are $b_1 c_2 - b_2 c_1$, $c_1 a_2 - c_2 a_1$, $a_1 b_2 - a_2 b_1$.

Putting $z = 0$ in (1) and (2), we get,

$$a_1 x + b_1 y + d_1 = 0$$
$$a_2 x + b_2 y + d_2 = 0$$

$$\therefore \quad \frac{x}{b_1d_2 - b_2d_1} = \frac{y}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \quad y = \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, \quad z = 0$$

$\therefore$ one point on the line of intersection of planes (1) and (2) is

$$\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right).$$

$\therefore$ the equations of the line of intersection of the planes (1) and (2) are

$$\frac{x - \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

If this line is parallel to the plane (3), then it is perpendicular to the normal to the plane (3) and its point

$\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right)$ does not lie in the plane (3).

$$\therefore \quad (b_1 c_2 - b_2 c_1) a_3 + (c_1 a_2 - c_2 a_1) b_3 + (a_1 b_2 - a_2 b_1) c_3 = 0$$

and $a_3 \left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}\right) + c_3(0) + d_3 \neq 0$

or $\quad a_3 (b_1 c_2 - b_2 c_1) - b_3 (c_2 a_1 - c_1 a_2) + c_3 (a_1 b_2 - a_2 b_1) = 0$

and $\quad a_3 (b_1 d_2 - b_2 d_1) - b_3 (d_2 a_1 - d_1 a_2) + d_3 (a_1 b_2 - a_2 b_1) \neq 0$

or $\quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ and $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \neq 0$

which are the required conditions.

Note. The three planes will form a triangular prism

if $\Delta = 0$ and $\Delta_1 \neq 0$ ...(1)

or if $\Delta = 0$ and $\Delta_2 \neq 0$ ...(2)

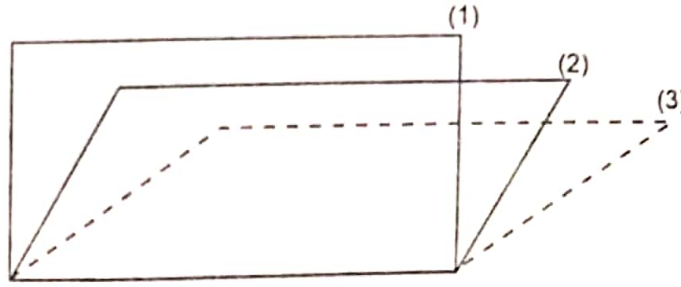
or if $\Delta = 0$ and $\Delta_3 \neq 0$ ...(3)

Case III. Find the condition that three given planes may have a common line of intersection.

Proof. Let the equations of the planes be

$$a_1 x + b_1 y + c_1 z + d_1 = 0 \quad \ldots(1)$$
$$a_2 x + b_2 y + c_2 z + d_2 = 0 \quad \ldots(2)$$
$$a_3 x + b_3 y + c_3 z + d_3 = 0 \quad \ldots(3)$$

If the three planes have a common line of intersection, then the line of intersection of any of two planes, say (1) and (2), lies in the third plane (3).



First of all, we find the equations of the line of intersection of planes (1) and (2) in the symmetrical form.

Omitting the constant terms from (1) and (2), we get,

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$

$$\therefore \quad \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

$\therefore$ d.e's. of the line of intersection of planes (1) and (2) are proportional to $b_1 c_2 - b_2 c_1$, $c_1 a_2 - c_2 a_1$, $a_1 b_2 - a_2 b_1$.

Putting $z = 0$ in (1) and (2), we get,

$$a_1 x + b_1 y + d_1 = 0$$
$$a_2 x + b_2 y + d_2 = 0$$

$$\therefore \quad \frac{x}{b_1d_2 - b_2d_1} = \frac{y}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \quad y = \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, \quad z = 0$$

$\therefore$ one point on the line of intersection of the planes (1) and (2) is

$$\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right)$$

$\therefore$ the equations of the line of intersection of the planes (1) and (2) are

$$\frac{x - \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

If it lies in the plane (3), then it is perpendicular to the normal to the plane (3) and its point $\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right)$ lies on the plane (3).

$$\therefore \quad (b_1 c_2 - b_2 c_1) a_3 + (c_1 a_2 - c_2 a_1) b_3 + (a_1 b_2 - a_2 b_1) c_3 = 0$$

and $$a_3\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}\right)+c_3(0)+d_3=0$$

or $$a_3(b_1c_2-b_2c_1)-b_3(c_2a_1-c_1a_2)+c_3(a_1b_2-a_2b_1)=0$$

and $$a_3(b_1d_2-b_2d_1)-b_3(d_2a_1-d_1a_2)+d_3(a_1b_2-a_2b_1)=0$$

or $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \text{ and } \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}=0$$

which are the required conditions.

Note 1. The three planes have a common line of intersection if $\Delta=\Delta_1=\Delta_2=\Delta_3=0$

Note 2. It is sufficient to prove that $\Delta=0$ and only any one of Δ_1, Δ_2, Δ_3 is zero.

Note 3. Another method for numerical examples

(*i*) Write down the equation of any plane through the line of intersection of the first two planes

i.e., first plane + k (second plane) = 0

(*ii*) Because three planes have a common line of intersection, therefore, for some value of k, this equation is the same as the equation of the third plane. Compare coeffs.

(*iii*) Find the value of k from the first and second members.

(*iv*) Substitute this value of k in all the members, and show that all the equations are satisfied.

ILLUSTRATIVE EXAMPLES

Example 1. Show that the planes $x-2y+z=0$; $x+y-2z-3=0$; $3x-2y+z-2=0$ meet in a point. Find the co-ordinates of the point.

Sol. The equations of the planes are

$$x-2y+z=0$$
$$x+y-2z=3$$
$$3x-2y+z=2$$

Here $$\Delta=\begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix}=1(1-4)+2(1+6)+1(-2-3)=-3+14-5=6\neq 0$$

$\therefore$ given planes meet in a point.

$$\Delta_1=\begin{vmatrix} 0 & -2 & 1 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}=2(3+4)+1(-6-2)=14-8=6$$

$$\Delta_2=\begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & -2 \\ 3 & 2 & 1 \end{vmatrix}=1(3+4)+1(2-9)=7-7=0$$

$$\Delta_3=\begin{vmatrix} 1 & -2 & 0 \\ 1 & 1 & 3 \\ 3 & -2 & 2 \end{vmatrix}=1(2+6)+2(2-9)=8-14=-6$$

Now $x = \frac{\Delta_1}{\Delta} = \frac{6}{6} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{0}{6} = 0$, $z = \frac{\Delta_3}{\Delta} = \frac{-6}{6} = -1$

$\therefore$ point is $(1, 0, -1)$

Example 2. Prove that the four planes

$$my + nz = 0, nz + lx = 0, lx + my = 0, lx + my + nz = p$$

form a tetrahedron whose volume is $\frac{2p^3}{3lmn}$.

Sol. The equations of given planes are

$$my + nz = 0 \quad \ldots(1)$$

$$nz + lx = 0 \quad \ldots(2)$$

$$lx + my = 0 \quad \ldots(3)$$

$$lx + my + nz = p \quad \ldots(4)$$

Adding (1), (2), (3), we get,

$$2(lx + my + nz) = 0, \quad \text{or} \quad lx + my + nz = 0$$

Subtracting (1), (2), (3) from this equation, we get,

$$lx = 0, \; my = 0, \; nz = 0$$

$\therefore \quad x = 0, \quad y = 0, \quad z = 0 \qquad [\because l \neq 0, m \neq 0, n \neq 0]$

$\therefore$ one vertex of the tetrahedron is $(0, 0, 0)$

Again (1) + (2) – (4) gives us

$$nz = -p \quad \text{or} \quad z = -\frac{p}{n}$$

Putting $z = -\frac{p}{n}$ in (1), we get,

$$my - p = 0 \quad \text{or} \quad y = \frac{p}{m}$$

Putting $z = -\frac{p}{n}$ in (2), we get,

$$n\left(-\frac{p}{n}\right) + lx = 0 \text{ or } x = \frac{p}{l}$$

$\therefore \left(\frac{p}{l}, \frac{p}{m}, -\frac{p}{n}\right)$ is the second vertex of the tetrahedron.

Similarly $\left(\frac{p}{l}, \frac{-p}{m}, \frac{p}{n}\right)$, $\left(\frac{-p}{l}, \frac{p}{m}, \frac{p}{n}\right)$ are the other two vertices of the tetrahedron

Now we know that

$$\text{Volume of the tetrahedron} = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ \frac{p}{l} & \frac{p}{m} & \frac{-p}{n} & 1 \\ \frac{p}{l} & \frac{-p}{m} & \frac{p}{n} & 1 \\ \frac{-p}{l} & \frac{p}{m} & \frac{p}{n} & 1 \end{vmatrix}$$

$$\left[\because \text{ volume} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}\right]$$

$$= -\frac{1}{6}\begin{vmatrix} \frac{p}{l} & \frac{p}{m} & -\frac{p}{n} \\ \frac{p}{l} & -\frac{p}{m} & \frac{p}{n} \\ -\frac{p}{l} & \frac{p}{m} & \frac{p}{n} \end{vmatrix}$$

$$= \left(-\frac{1}{6}\right)\left(\frac{p}{l}\right)\left(\frac{p}{m}\right)\left(\frac{p}{n}\right)\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= -\frac{1}{6}\frac{p^3}{l\,m\,n}\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ -1 & 2 & 0 \end{vmatrix}, \text{ by } C_2 - C_1,\ C_3 + C_1$$

$$= -\frac{p^3}{6lmn}\begin{vmatrix} -2 & 2 \\ 2 & 0 \end{vmatrix} = -\frac{p^2}{6lmn}(0-4)$$

$$= \frac{2\,p^3}{3\,l\,m\,n}.$$

Note. Rule to find the area of a normal section of a triangular prism.

We are given three planes (1), (2), (3).

(*i*) Find the *d. cs.* of the normal to the normal section *i.e.*, the *d. cs.* of the line of intersection of the planes (1) and (2).

(*ii*) Put $z = 0$ in (1), (2), (3) to get the equations of the sides of the triangle, which is the section of the triangular prism by the plane $z = 0$.

(*iii*) Find Δ, the area of this triangle.

(*iv*) Area of normal section is given by $\Delta' = \Delta \cos\alpha$ where α is the angle between *xy*-plane and the normal section *i.e.*, the angle between the *z*-axis and the normal to the normal section.

Example 3. Show that the planes $2x + 3y + 4z = 6$, $3x + 4y + 5z = 2$, $x + 2y + 3z = 2$ form a prism and find area of its normal section.

(G.N.D.U. 2008)

Sol. The equations of planes are

$$2x + 3y + 4z = 6 \quad \ldots(1)$$
$$3x + 4y + 5z = 2 \quad \ldots(2)$$
$$x + 2y + 3z = 2 \quad \ldots(3)$$

Now
$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$
$$= 2(12 - 10) - 3(9 - 5) + 4(6 - 4)$$
$$= 2(2) - 3(4) + 4(2) = 4 - 12 + 8 = 0$$

$$\Delta_1 = \begin{vmatrix} 6 & 3 & 4 \\ 2 & 4 & 5 \\ 2 & 2 & 3 \end{vmatrix} = 6\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix}$$
$$= 6(12 - 10) - 3(6 - 10) + 4(4 - 8)$$
$$= 6(2) - 3(-4) + 4(-4)$$
$$= 12 + 12 + 16 = 40 \neq 0$$

$\therefore$ given planes form a prism.

Now we will find the d.cs. of the normal to the normal section *i.e.* the d.cs. of the line of intersection of the planes (1) and (2). Omitting constant terms in (1) and (2), we get,

$$2x + 3y + 4z = 0$$
$$3x + 4y + 5z = 0$$

$$\therefore \quad \frac{x}{15-16} = \frac{y}{12-10} = \frac{z}{8-9} \quad \Rightarrow \quad \frac{x}{-1} = \frac{y}{2} = \frac{z}{-1}$$

$\therefore$ direction-ratios of the line are $< -1, 2, -1 >$

$\therefore$ d.cs. of the normal to the normal section are $\left\langle -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$.

Also d.cs. of the z-axis are 0, 0, 1

Let α be the angle between the xy-plane and the normal to the normal section *i.e.* the angle between the z-axis and the normal to the normal section.

$$\therefore \quad \cos\alpha = \left| (0)\left(-\frac{1}{\sqrt{6}}\right) + (0)\left(\frac{2}{\sqrt{6}}\right) + (1)\left(-\frac{1}{\sqrt{6}}\right) \right| = \frac{1}{\sqrt{6}}$$

Putting $z = 0$ in (1), (2), (3), we get,

$$2x + 3y - 6 = 0 \quad \ldots(4)$$
$$3x + 4y - 2 = 0 \quad \ldots(5)$$
$$x + 2y = 0$$

Solving (4) and (5), we get,

$$\frac{x}{-6+24} = \frac{y}{-18+4} = \frac{1}{8-9} \quad \text{or} \quad \frac{x}{18} = \frac{y}{-14} = \frac{1}{-1}$$

$\therefore \qquad x = -18,\ y = 14$

Solving (5) and (6), we get

$$\frac{x}{0+4} = \frac{y}{-2-0} = \frac{1}{6-4} \qquad \text{or} \qquad \frac{x}{4} = \frac{y}{-2} = \frac{1}{2}$$

$\therefore \qquad x = 2,\ y = -1$

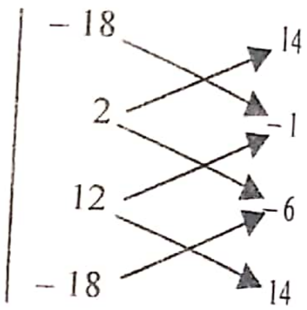
Solving (4) and (6), we get,

$$\frac{x}{0+12} = \frac{y}{-6-0} = \frac{1}{4-3} \qquad \text{or} \qquad \frac{x}{12} = \frac{y}{-6} = \frac{1}{1}$$

$\therefore \qquad x = 12,\ y = -6$

$\therefore$ vertices of the triangle (the section of the triangular prism by the plane $z = 0$) are $(-18, 14), (2, -1), (12, -6)$

Let Δ be the area of this triangle.



$$\therefore \qquad \Delta = \frac{1}{2}\,|\,18 - 28 - 12 + 12 + 168 - 108\,|$$

$$= \frac{1}{2}\,|\,50\,| = 25$$

$\therefore$ area of the normal section is

$$\Delta' = \Delta \cos\alpha = 25 \times \frac{1}{\sqrt{6}}$$

$$= \frac{25}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{25\sqrt{6}}{6} \text{ sq. units.}$$

Example 4. Prove that the planes $x = cy + bz,\ y = az + cx,\ z = bx + ay$ pass through one line if $a^2 + b^2 + c^2 + 2abc = 1$. Show that the equations of this line are

$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}.$$

(G.N.D.U. 2007, 2008, 2009)

Sol. The equations of the planes are

$$x - cy - bz = 0$$
$$cx - y + az = 0$$
$$bx + ay - z = 0$$

$$\Delta = \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 1(1 - a^2) + c(-c - ab) - b(ca + b)$$

$$= 1 - a^2 - c^2 - abc - abc - b^2 = 1 - a^2 - b^2 - c^2 - 2abc$$

$$\Delta_1 = \begin{vmatrix} 0 & -c & -b \\ 0 & -1 & a \\ 0 & a & -1 \end{vmatrix} = 0 \text{ as first column has got each element as zero}$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & -b \\ c & 0 & a \\ b & 0 & -1 \end{vmatrix} = 0$$

$$\Delta_3 = \begin{vmatrix} 1 & -c & 0 \\ c & -1 & 0 \\ b & a & 0 \end{vmatrix} = 0$$

We know that three given planes will intersect in a line

if $\Delta = 0, \quad \Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0$

i.e., if $1 - a^2 - b^2 - c^2 - 2abc = 0$

i.e., if $a^2 + b^2 + c^2 + 2abc = 1$ [$\because \Delta_1 = \Delta_2 = \Delta_3 = 0$ are satisfied]

which is the required condition.

Now required line of intersection is the line of intersection of (1) and (2).

Omitting constant terms in (1) and (2), we get,

$$x - cy - bz = 0$$
$$cx - y + az = 0$$

$$\therefore \quad \frac{x}{-ac-b} = \frac{y}{-bc-a} = \frac{z}{-1+c^2}$$

$\therefore$ $d.r s$ of the line are $-ac-b, -bc-a, -1+c^2$

or $\sqrt{(-ac-b)^2}, \sqrt{(-bc-a)^2}, \sqrt{(-1+c^2)^2}$

or $\sqrt{a^2c^2+b^2+2abc}, \sqrt{b^2c^2+a^2+2abc}, \sqrt{(1-c^2)^2}$

or $\sqrt{a^2c^2+1-a^2-c^2}, \sqrt{b^2c^2+1-b^2-c^2}, \sqrt{(1-c^2)^2}$

or $\sqrt{(1-c^2)(1-a^2)}, \sqrt{(1-c^2)(1-b^2)}, \sqrt{(1-c^2)^2}$

or $\sqrt{1-a^2}, \sqrt{1-b^2}, \sqrt{1-c^2}$

Putting $z = 0$ in (10 and (2), we get,

$$x - cy = 0$$
$$cx - y = 0$$

Solving these, we get, $x = 0, y = 0$

$\therefore$ line of intersection of planes (1) and (2) (1) passes through the point (0, 0, 0)

$\therefore$ equations of the line of intersection are

$$\frac{x-0}{\sqrt{1-a^2}} = \frac{y-0}{\sqrt{1-b^2}} = \frac{z-0}{\sqrt{1-c^2}} \quad \text{or} \quad \frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$$

EXERCISE 3 (*a*)

1. Prove that the planes $2x - y + z - 4 = 0$, $5x + 7y + 2z = 0$ and $3x + 4y - 2z + 3 = 0$ meet in a point. Find the co-ordinates of their point of intersection.

2. (*i*) Find the point of intersection of the planes
$x - y + 2z = 3$, $x + 2y + 3z = 5$, $3x - 4y - 5z + 13 = 0$

(G.N.D.U. 2004)

(*ii*) Show that the planes $x + y + z = 6$, $2x + 3y + 4z = 20$, $x - y + z = 2$ meet in a point. Find the coordinates of the point.

(Pbi. U. 2004)