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MOMENTUM AND IMPULSE

Art-1. Linear Momentum

If m is the mass of a particle moving with velocity $\vec{v}$, then its linear momentum is the product $m\vec{v}$ and is generally denoted by $\vec{p}$

$$\therefore \qquad \vec{p} = m\vec{v}$$

Let (x, y) be the position of the particle at any time t.

$$\therefore \qquad \vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\therefore \qquad \vec{p} = m\vec{v} \quad \Rightarrow \quad \vec{p} = \left(m\frac{dx}{dt}\right)\hat{i} + \left(m\frac{dy}{dt}\right)\hat{j}$$

$\therefore$ components of momentum are $m\dfrac{dx}{dt}$ and $m\dfrac{dy}{dt}$. They are generally denoted by p_x and p_y.

$$\therefore \qquad p_x = m\frac{dx}{dt} \quad \text{and} \quad p_y = m\frac{dy}{dt}$$

Art-2. Conservation of Linear Momentum

We have $\qquad \vec{p} = m\vec{v}$ ...(1)

According to Newton's second law of motion, the rate of change of momentum is equal to the applied force $\vec{F}$

i.e. $$\frac{d\vec{p}}{dt} = \vec{F}$$

If $\vec{F} = \vec{0}$ *i.e.* no force acts on the particle, then

$$\frac{d\vec{p}}{dt} = \vec{0}$$

Integrating w.r.t. t, we get,

$$\vec{p} = \text{constant vector}$$

or $\qquad m\vec{v} = \text{constant vector}$ [$\because$ of (1)]

$\Rightarrow$ *the linear momentum of a particle remains constant in the absence of an applied force.* This is known as the **principle of conservation of linear momentum.**

Art-3. Angular Momentum

We know that linear momentum is a vector quantity and so it can have a moment about any point in the plane.

The moment of linear momentum about a point in the plane is called the moment of momentum or angular momentum.

If m is the mass and v is the velocity of the particle, the angular momentum J of the linear momentum $m\,v$ about a point O at a distance r in the plane of O is given by

$$J = m\,v\,r \qquad ...(1)$$

The direction of J is perpendicular to the plane.

Also we know that the moment of a vector about a point is equal to the algebraic sum of the moments of its components about the same point.

$\therefore$ the magnitude of the angular momentum of a particle about the origin O is given by

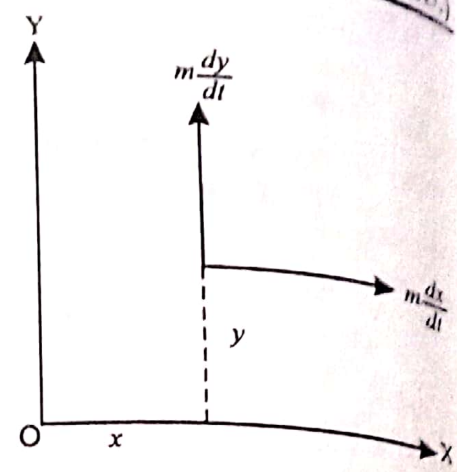
$$J = -m\frac{dx}{dt}\cdot y + m\frac{dy}{dt}\cdot x$$

or

$$J = m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \qquad ...(2)$$

where m is the mass of the particle.

When the momentum vector is resolved along and perpendicular to the radius vector drawn from the origin, then



p_r = momentum along radius vector $= m\dfrac{dr}{dt}$.

[∵ velocity along radius vector $= \dfrac{dr}{dt}$]

and p_θ = momentum perpendicular to radius vector

$$= m\,r\,\frac{d\theta}{dt}$$

[∵ velocity perp. to radius vector $= r\dfrac{d\theta}{dt}$]

Now angular momentum along the radius vector about O = 0, as radius vector passes through O.

∴ the angular momentum perp. to radius vector $= \left(m\,r\dfrac{d\theta}{dt}\right) r = m\,r^2\dfrac{d\theta}{dt}$

$$\therefore \qquad J = m\,r^2\,\frac{d\theta}{dt} \qquad ...(3)$$

The units of angular momentum are obtained by multiplying those of linear momentum with units of distance.

Principal of Angular Momentum

Let P (x, y) be the position of the particle of mass m, which moves under the action of the force F with components X and Y, at time t.

Therefore the equations of motion are

$$X = m\frac{d^2x}{dt^2} \text{ and } Y = m\frac{d^2y}{dt^2}$$

Now
$$J = m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

$$\therefore \qquad \frac{dJ}{dt} = m\left(x\frac{d^2y}{dt^2} + \frac{dx}{dt}\frac{dy}{dt} - \frac{dy}{dt}\frac{dx}{dt} - y\frac{d^2x}{dt^2}\right)$$

$$= m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = x\left(m\frac{d^2y}{dt^2}\right) - y\left(m\frac{d^2x}{dt^2}\right)$$

$$= x\,Y - y\,X = N$$

where N is the moment of force F about the origin.

This result is known as the Principle of Angular Momentum.

Since any point in the plane can be taken as origin.

∴ Principle of Angular Momentum is

The rate of change of angular momentum of a particle moving in a plane about a fixed point is equal to the moment of the force about the point.

Principle of Conservation of Angular Momentum

We have $\frac{dJ}{dt} = xY - yX = N$ = Moment of the force about the origin

When N = 0, then $\frac{dJ}{dt} = 0 \Rightarrow J = \text{constant}$

We know that the moment of a force about origin O vanishes if either the force is zero or the line of action of the force passes through the origin.

Thus we arrive at Principle of conservation of Angular Momentum which may be stated as :

If a particle is in motion in a plane and there is no force acting on the particle or the line of action of the force passes through the origin, then the angular momentum about the origin remains constant.

Art-4. Impulse of a Force

(a) **Impulse of a constant force :** *When a force is constant in magnitude and direction, the impulse is the product of the force and the time during which it acts.*

If a constant force F, acting on a particle of mass m, produces an acceleration f and changes the velocity of the particle from u to v in time t, then

$$\text{impulse } P = F\,t = (m f)\,t = m\,(f t)$$

$$= m\,(v-u) \qquad [\because\ v = u + f t]$$

$$= m\,v - m\,u$$

$$= \text{change in momentum}$$

$\therefore$ impulse = change of linear momentum in the direction of the force.

If $u = 0$ i.e. the particle starts from rest, then $P = m\,v - 0 = m\,v$

$\therefore$ impulse = momentum generated in time t

or force F = momentum generated per unit time.

(b) **Impulse of a variable force :** Let a variable force act on a particle of mass m for time T. Let F be the force at time t, then the impulse of the force is defined as the integral $\int_0^T F\,dt$

Now $$F = \frac{d}{dt}(m\,v) \qquad [\because \text{ of Newton's second law}]$$

where v is the velocity of the particle at time t.

$$\therefore \quad F = m\frac{dv}{dt} \Rightarrow F\,dt = m\,dv$$

$\therefore$ the whole impulse in time T during which the velocity of the particle changes from u to v.

$$= \int_u^v m\,dv = m\,(v-u)$$

$$= m\,v - m\,u$$

$$= \text{Change in linear momentum}$$

$\therefore$ impulse = change of linear momentum in the direction of the force.

Hence the impulse is the change of linear momentum whether the force is constant or variable.

Note. Impulse is not a force. It is force × time and equals change in momentum.

Art-5. Impulsive Force

Impulsive force is that which acts for a very short time but is great in magnitude.

Examples :

(*i*) Blow of a hammer

(*ii*) Tension in a string produced by a jerk

Art-6. Motion With Respect to the Centre of Mass of System of Particles

Let (x_1, y_1), (x_2, y_2), ... be the co-ordinates of a number of coplanar particles of masses m_1, m_2, ... Let $(\bar{x}, \bar{y})$ be the co-ordinates of their centre of mass. Then

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \dots}{m_1 + m_2 + \dots}$$

and

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + \dots}{m_1 + m_2 + \dots}$$

$\therefore$ $$M\bar{x} = m_1 x_1 + m_2 x_2 + ... \quad ...(1)$$

and $$M\bar{y} = m_1 y_1 + m_2 y_2 + ... \quad ...(2)$$

where $M = m_1 + m_2 + ...$ is the sum of masses.

Differentiating (1) and (2) w.r.t. t, we get,

$$M\frac{d\bar{x}}{dt} = m_1\frac{dx_1}{dt} + m_2\frac{dx_2}{dt} + ... \quad ...(3)$$

and $$M\frac{d\bar{y}}{dt} = m_1\frac{dy_1}{dt} + m_2\frac{dy_2}{dt} + ... \quad ...(4)$$

From (3) and (4), it is clear that

The total momentum, in any direction, of the particles is the same as that of a particle whose mass is the whole mass and whose velocity is equal to the velocity of the centre of gravity of the particles.

Differentiating (3) and (4) w.r.t. t, we get,

$$M\frac{d^2\bar{x}}{dt^2} = m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} + ... \quad ...(5)$$

and $$M\frac{d^2\bar{y}}{dt^2} = m_1\frac{d^2y_1}{dt^2} + m_2\frac{d^2y_2}{dt^2} + ... \quad ...(6)$$

These equations (5) and (6) give the acceleration of the centre of mass of the particles.

Let the particles be subjected to given external forces whose components parallel to the axes are $(X_1, Y_1), (X_2, Y_2), ...$

Since the internal forces, being of the nature of action and reaction, cancel each other

$\therefore$ equations (5) and (6) can be written as

$$M\frac{d^2\bar{x}}{dt^2} = X_1 + X_2 + ...$$

and $$M\frac{d^2\bar{y}}{dt^2} = Y_1 + Y_2 + ...$$

$\therefore$ *the motion of the centre of gravity of the particles in any direction is the same as that of a mass equal to the whole mass of the particles and acted on the by the forces which actually act on the separate particles.*

Note : When there is no external force acting on the system of particles, the momentum of their centre of mass remains constant and so the velocity of the centre of mass is constant.

Art-7. Collision of Elastic Bodies

We have discussed the impact of two inelastic bodies (i.e. bodies which do not rebound after impact). Now we shall discuss the impact of two elastic bodies (i.e. bodies which rebound after impact).

There are two types of impact between two elastic bodies.

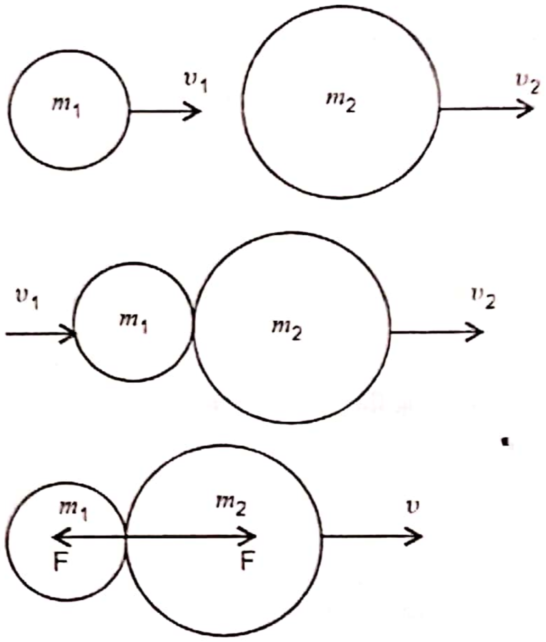
(*i*) **Direct Impact.** *If the two bodies collide so that the direction of each is along the common normal at the point of contact, the impact is said to be direct.*

(*ii*) **Oblique Impact.** *If the two bodies collide so that the direction of either or both is not along the common normal at the point of contact, the impact is said to be oblique.*

Art-8. Direct Impact of Two Smooth Spheres

Let the two smooth spheres of masses m_1 and m_2 moving in straight line with velocities v_1 and v_2 collide each other. Find the velocities after impact.

Proof : Initially the spheres are moving with velocity v_1 and v_2 both being measured positive to the right.



When the two spheres come into contact, deformation takes place and reactional forces come into existence. These forces are very large and exist for a very short time as long as the spheres remain in contact. Moreover, these reactional forces act along the common normal i.e. the line of centres and tend to push two spheres apart.

After contact, the spheres compress each other till a stage arrives when the compression is greatest, which happens for a very short time τ, say.

Let v be the common velocity of two spheres when the compression is maximum. Then, for the two spheres, we have,

$$m_1 v - m_1 v_1 = \int_0^{\tau} (-F)\, dt = -P \qquad ...(1)$$

and $$m_2 v - m_2 v_2 = \int_0^{\tau} F\, dt = P \qquad ...(2)$$

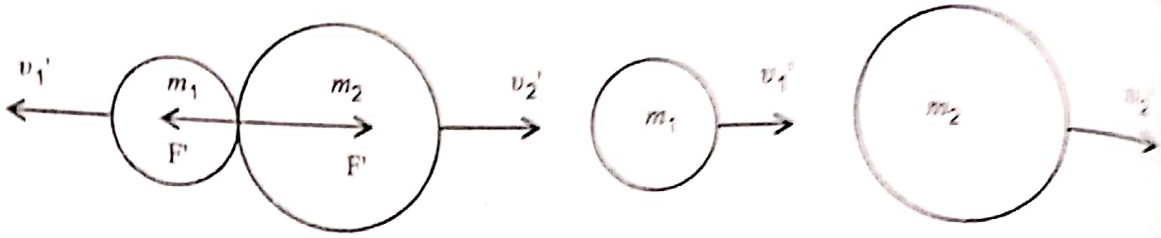
where P is the magnitude of the impulse of the force acting on the sphere of mass m_2.

From (1), $m_1 v_1 = m_1 v + P \Rightarrow v_1 = v + \dfrac{P}{m_1}$ ...(3)

From (2), $m_2 v_2 = m_2 v - P \Rightarrow v_2 = v - \dfrac{P}{m_2}$ ...(4)

Subtracting (4) from (3), we get,

$$v_1 - v_2 = P\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \Rightarrow v_1 - v_2 = P\left(\frac{m_1 + m_2}{m_1 m_2}\right) \quad ...(5)$$



After the instant of maximum compression, the period of restitution begins and another system of forces comes into existence which helps the two spheres recover their original shape. These forces are also large and act for a very small period τ', say. Let v_1', v_2' be the velocities of the two spheres just after the impact.

$$\therefore \quad m_1 v_1' - m_1 v = \int_0^{\tau'} (-F')\, dt = -P' \quad ...(6)$$

and

$$m_2 v_2' - m_2 v = \int_0^{\tau'} F'\, dt = P' \quad ...(7)$$

where P' is the magnitude of the impulse of the force acting on the sphere of mass m_2.

From (6), $m_1 v_1' = m_1 v - P' \Rightarrow v_1' = v - \dfrac{P'}{m_1}$ ...(8)

From (7), $m_2 v_2' = m_2 v + P' \Rightarrow v_2' = v + \dfrac{P'}{m_2}$ ...(9)

Subtracting (9) from (8), we get,

$$v_1' - v_2' = -P'\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$\Rightarrow \quad v_1' - v_2' = -P'\left(\frac{m_1 + m_2}{m_1 m_2}\right) \quad ...(10)$$

P and P′ are the impulses of compression and impulse of restitution respectively. It has been proved that for every pair of colliding objects, there exists a positive constant e $(0 \le e \le 1)$ such that

$$P' = e P \quad ...(11)$$

Dividing (5) by (10), we get,

$$\frac{v_1 - v_2}{v'_1 - v'_2} = -\frac{P}{P'} \quad \text{or} \quad \frac{v_1 - v_2}{v'_1 - v'_2} = -\frac{1}{e} \qquad [\because \text{ of } (11)]$$

$$\Rightarrow \qquad v'_1 - v'_2 = -e\,(v_1 - v_2) \qquad ...(12)$$

This result shows that *the relative velocity of the two spheres after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction. This is known as* **Newton's experimental law** and the constant e is called the **coefficient of restitution (or elasticity).**

By the principle of conservation of linear momentum,

$$m_1 v'_1 + m_2 v'_2 = m_1 v_1 + m_2 v_2 \qquad ...(13)$$

$m_2 \times (12) + (13)$ gives us

$$m_2 v'_1 + m_1 v'_1 = -m_2 e\,(v_1 - v_2) + m_1 v_1 + m_2 v_2$$

$$\therefore \qquad v'_1 = \frac{m_1 v_1 + m_2 v_2 - e m_2 (v_1 - v_2)}{m_1 + m_2} \qquad ...(14)$$

$(13) - m_1 \times (12)$ gives us

$$m_2 v'_2 + m_1 v'_2 = m_1 v_1 + m_2 v_2 + e\, m_1 (v_1 - v_2)$$

$$\therefore \qquad v'_2 = \frac{m_1 v_1 + m_2 v_2 + e m_1 (v_1 - v_2)}{m_1 + m_2} \qquad ...(15)$$

From (14) and (15), we get the velocities after impact.

Note : For perfectly elastic bodies $e = 1$ and for inelastic bodies $e = 0$

Cor 1. *When the bodies are perfectly inelastic,*

then $\qquad e = 0$

$\therefore$ from (12), $v'_1 - v'_2 = 0 \Rightarrow v'_1 = v'_2$

$\therefore$ two bodies move together with the same velocity after impact.

From (10), $-P'\left[\dfrac{m_1 + m_2}{m_1 m_2}\right] = 0 \Rightarrow P' = 0$

$\therefore$ from (8), $\qquad v'_1 = v$

and from (9), $\qquad v'_2 = v$

$$\therefore \qquad v'_1 = v'_2 = v$$

$\therefore$ two bodies remain in contact with maximum deformation travelling with velocity v

Cor 2. *When the bodies are perfectly elastic,*

then $e = 1$

$\therefore$ from (12), $v_1' - v_2' = -(v_1 - v_2)$

$\therefore$ relative velocity after impact is the same as that before impact but in the opposite direction.

Art-9. Loss of Energy During Impact

Let m_1, m_2 be the masses of two spheres. Let v_1, v_2 be the velocities of the two spheres before impact and v_1', v_2' be the velocities after impact.

By the principle of conservation of linear momentum ;

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad ...(1)$$

By Newton's experimental law,

$$v_1' - v_2' = -e(v_1 - v_2) \qquad ...(2)$$

where e is the co-efficient of restitution

$$\text{Change in kinetic energy} = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 - \left(\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2\right)$$

$$= \frac{1}{2}\left(\frac{m_1 + m_2}{m_1 + m_2}\right)[m_1 v_1'^2 + m_2 v_2'^2 - m_1 v_1^2 - m_2 v_2^2]$$

$$= \frac{1}{2(m_1 + m_2)}[(m_1 + m_2)(m_1 v_1'^2 + m_2 v_2'^2) - (m_1 + m_2)(m_1 v_1^2 + m_2 v_2^2)]$$

$$= \frac{1}{2(m_1 + m_2)}[m_1 m_2 (v_1' - v_2')^2 + (m_1 v_1' + m_2 v_2')^2 - m_1 m_2 (v_1 - v_2)^2 - (m_1 v_1 + m_2 v_2)^2]$$

$$= \frac{m_1 m_2}{2(m_1 + m_2)}[(v_1' - v_2')^2 - (v_1 - v_2)^2] \qquad [\because \text{ of } (1)]$$

$$= \frac{m_1 m_2}{2(m_1 + m_2)}[e^2 (v_1 + v_2)^2 - (v_1 - v_2)^2] \qquad [\because \text{ of } (2)]$$

$$= \frac{-m_1 m_2}{2(m_1 + m_2)}(v_1 - v_2)^2 (1 - e^2).$$

Since $0 < e < 1$ (in general), the change in kinetic energy is negative. Hence, there is a loss of kinetic energy during impact and this loss equals $\frac{m_1 m_2 (1 - e^2)}{2(m_1 + m_2)}(v_1 - v_2)^2$.