6

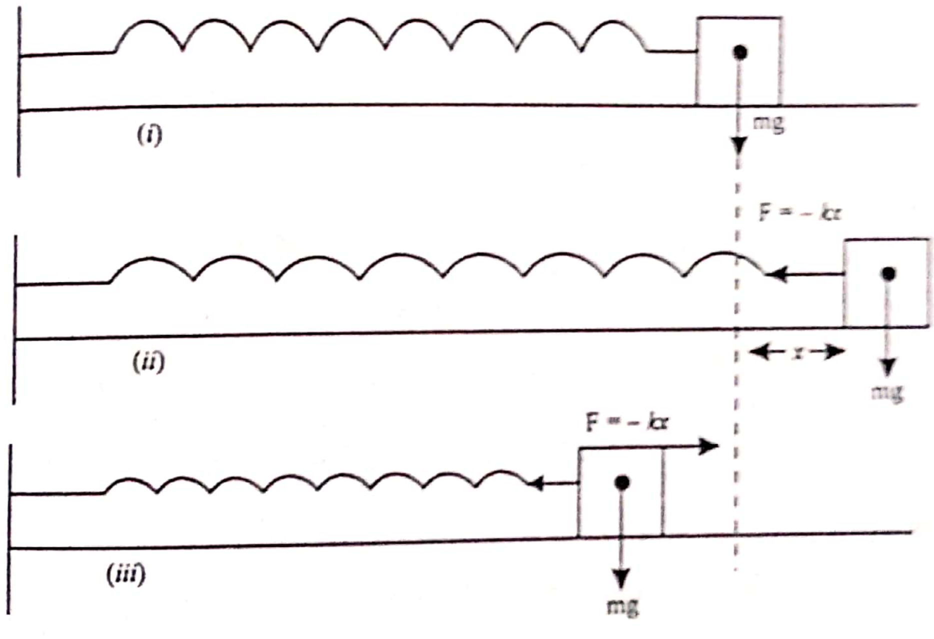
OSCILLATIONS

Art-1. Introduction

We have already discussed simple harmonic motion, harmonic motion and periodic motion. In many machines, there are oscillating (vibrating) parts and to these parts we have to apply periodic forces. In this chapter, we will study some cases of vibrating motions.

Art-2. Free Vibrations

Most of the problems of vibrating motion can be discussed by a single mathematical model called the **harmonic oscillator.** It consists of a particle of mass m which is subjected to a force, which varies linearly as the distance and directed towards the fixed point.



Let a body of mass m be attached to an ideal spring of force constant k, which is free to move over a frictionless horizontal surface.

The spring exerts no force on the particle in the position of equilibrium as in figure (i).

When the particle is displaced to the right by a distance x as in figure (ii), the spring exerts a force F to the left and is give by

$$F = -kx$$

When the particle is displaced to the left by a distance x as in figure (*iii*), the spring exerts a force F to the right and is give by

$$F = -kx$$

Thus, in each case, the force acting on the particle is a restoring one.

The equation of motion of the harmonic oscillator is

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad m\frac{d^2x}{dt^2} + kx = 0$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + \omega_0^2 x = 0 \qquad ...(1)$$

where $\omega_0^2 = \frac{k}{m}$

∴ the motion of the harmonic oscillator is S.H.M. The constant ω_0 is called the natural frequency of the oscillator.

In equation (1) S.F. is $\left(D^2 + \omega_0^2\right)x = 0$

A.E. is $D^2 + \omega_0^2 = 0$

∴ $D^2 = -\omega_0^2 \Rightarrow D = \pm i\omega_0 = 0 \pm i\omega_0$

∴ complete solution of (1) is

$$x = A\cos\omega_0 t + B\sin\omega_0 t \qquad ....(2)$$

Where A, B are constants of integration.

Let $A = A_0\cos\phi$, $B = A_0\sin\phi$

∴ $A^2 + B^2 = A_0{}^2 \Rightarrow A_0 = \sqrt{A^2+B^2}$

and $\cos\phi = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\phi = \frac{B}{\sqrt{A^2+B^2}}$

Putting these values of A and B in (2), we get

$$x = A_0\cos\phi\cos\omega_0 t + A_0\sin\phi\sin\omega_0 t$$

or $x = A_0\cos(\omega_0 t - \phi)$ ...(3)

Since $-1 \le \cos(\omega_0 t - \phi) \le 1$

∴ x lies between $-A_0$ and A_0.

∴ the motion is periodic with period $T = \frac{2\pi}{\omega_0}$.

Here A_0 is called amplitude, T is called period, $\frac{1}{T}$ is called frequency and $\omega_0 t - \phi$ is called phase of the harmonic oscillator.

ANSWERS

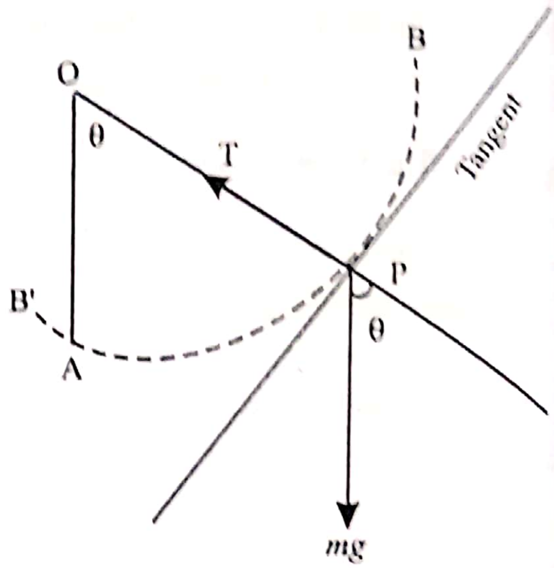
1. (*i*) 50 gm/sec (*ii*) 750 gms. wt. (*iii*) $\frac{2\pi}{\sqrt{70}}$ sec.

2. (*i*) 50 gm/sec (*ii*) 750 gms. wt. (*iii*) $\frac{2\pi}{\sqrt{50}}$ sec.

Art-3. Simple Pendulum

Definition. If a heavy particle, tied to one end of a light inextensible string, the other end of which is fixed, oscillates in a vertical circle, the system is called a simple pendulum.

Let O be the fixed point, l the length of the string, A the lowest position of the particle of mass m. Let P be the position of the particle at any time t such that $\angle AOP = \theta$ (radians). Here θ is the small angular displacement in a vertical plane.



The forces acting on the particle are

(*i*) its weight mg acting vertically downwards.

(*ii*) the tension T in the string along PO.

The equation of motion of the particle along the tangent to the circle at P is

$$m.l\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

[$\because$ acceleration along the tangent to a circle of radius l is $l\frac{d^2\theta}{dt^2}$]

or $l\frac{d^2\theta}{dt^2} = -g\sin\theta$ or $l\frac{d^2\theta}{dt^2} = -g\theta$ [$\because \sin\theta = \theta$ as θ is small]

or $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$

This equation shows that the motion is simple harmonic and the time period

$$T = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}$$

Note 1. The length l of the string is called the length of the simple pendulum.

Note 2. The time period depends upon l and the value of g at a place but is independent of θ, provided it is small.

Note 3. If the equation of motion is of the form $\frac{d^2x}{dt^2} = -\frac{g}{\lambda}x$, then the motion is oscillatory and the period of oscillation is the same as that of simple pendulum of length λ. Then λ is called the length of equivalent simple pendulum.

Note 4. Second Pendulum

Let B′, B be the extremities of the path of the simple pendulum.

$\therefore$ time form B to B′ and back of B $= 2\pi\sqrt{\frac{l}{g}}$

$\therefore$ time from B to B′ $= \frac{1}{2} \cdot 2\pi\sqrt{\frac{l}{g}} = \pi\sqrt{\frac{l}{g}}$

We know that pendulum of a clock beats at each extremity.

$\therefore$ time between two consecutive beats $= \pi\sqrt{\frac{l}{g}}$

A simple pendulum is called a second pendulum if the time interval between two consecutive beats is one second.

Let l be the correct length of a second's pendulum

$$\therefore \quad \pi\sqrt{\frac{l}{g}} = 1 \quad \Rightarrow \quad \pi^2\frac{l}{g} = 1 \quad \Rightarrow \quad l = \frac{g}{\pi^2}$$

In C.G.S. system, $g = 981$ cm/sec^2, $\pi^2 = 9.87$

$$\therefore \quad l = \frac{981}{9.87} = 9.9 \text{ cm}$$

In M.K.S. system, $g = 9.8$, $\pi^2 = 9.87$

$$\therefore \quad l = \frac{9.8}{9.87} = .992 \text{ metres}$$

Art-4. Formula for the Change in Number of Beats

Let n be the number of beats in a given time interval T

$\therefore$ T $= n \times$ time of one beat

or $T = n \times \pi\sqrt{\frac{l}{g}}$

$$\therefore \quad n = \frac{T}{\pi}\sqrt{\frac{g}{l}}$$

From (1), $\log n = \log\left(\frac{T}{\pi}\sqrt{\frac{g}{l}}\right)$

$$\therefore \quad \log n = \log\left(\frac{T}{\pi}\right) + \frac{1}{2}(\log g - \log l)$$

Taking differentials of both sides, we get,

$$\frac{1}{n}dn = \frac{1}{2}\left(\frac{1}{g}dg - \frac{1}{l}dl\right) \qquad \left[\because \frac{T}{\pi} \text{ is constant}\right]$$

$$\therefore \quad dn = \frac{n}{2}\left(\frac{1}{g}dg - \frac{1}{l}dl\right) \qquad \ldots(2)$$

where dn, dg, dl represent small changes in n, g, l respectively.

From (2), we get the number of beats lost or gained corresponding to the small variations in the values of length and gravity.

Particular Cases

(*i*) When only *l* changes and *g* remains fixed

$\therefore \quad dg = 0$

$$\therefore \quad \text{from (2), } dn = -\frac{n}{2}\cdot\frac{1}{l}dl \qquad \ldots(3)$$

If dl is positive, then from (3), dn is negative

i.e. if the length of the pendulum is increased and g remains same, then there is a decrease in the number of beats in any given time interval and hence the clock becomes slow.

Similarly if dl is negative, then form (3), dn is positive i.e. if the length of the pendulum is shortened and g remains same, then there is an increase in the number of beats in any given time interval and hence the clock runs fast.

(*ii*) When only *g* changes and *l* remains fixed.

$\therefore \quad dl = 0$

$$\therefore \quad \text{from (2), } dn = \frac{n}{2}\cdot\frac{1}{g}dg \qquad \ldots(4)$$

$\therefore$ dn is positive if dg is positive i.e. if g increases, there is an increase in the number of beats in any given time interval and hence the clock runs fast. Similarly if g decreases, then there is a decrease in the number of beats in any given time interval and hence the clock becomes slow.

Art-5. Change in Value of *g* and Number of Beats

(*i*) *When the pendulum is carried to a mountain at height h above the surface of earth.*

We know that, by Newton's gravitational law,

$g = \frac{\mu}{r^2}$ where r is the distance of the point outside the surface of earth from the centre of the earth and μ is constant.

$\therefore \quad \log g = \log \mu - 2 \log r$

$$\Rightarrow \quad \frac{1}{g}dg = 0 - \frac{2}{r}dr \qquad \Rightarrow \quad \frac{1}{g}dg = -\frac{2}{r}dr$$

Here $dr = h$

$$\therefore \quad \frac{1}{g}dg = -2\frac{h}{r} \qquad \text{...(1)}$$

Now $dn = \frac{n}{2}\frac{dg}{g}$

$$\therefore \quad dn = \frac{n}{2}\left(-\frac{2h}{r}\right) \qquad [\because \text{ of } (1)]$$

$$\therefore \quad dn = -\frac{nh}{r}$$

This equation shows that when a pendulum is carried to a high mountain, it loses number of beats in any given time interval and so the clock becomes slow.

(*ii*) *When the pendulum is carried inside a mine at a depth h below the surface of earth.*

We know that for the points inside the surface of the earth, $g = \lambda r$

where r is the distance of the point from the centre of the earth and λ is constant.

$$\therefore \quad \log g = \log \lambda + \log r$$

$$\Rightarrow \quad \frac{1}{g}dg = 0 + \frac{1}{r}dr \qquad \Rightarrow \frac{1}{g}dg = \frac{1}{r}dr$$

Here $\quad dr = -h$

$$\therefore \quad \frac{1}{g}dg = -\frac{h}{r} \qquad \text{...(2)}$$

Now $dn = \frac{n}{2}\frac{dg}{g} \quad \therefore \quad dn = \frac{n}{2}\left(-\frac{h}{r}\right) \qquad [\because \text{ of } (2)]$

$$\therefore \quad dn = -\frac{nh}{2r}$$

This equation shows that when a pendulum is carried to a deep mine, it loses number of beats in any given time interval and so the clock becomes slow.

Note 1. From above discussion, it is clear that *the pendulum, always becomes slow whether it is carried to a high mountain or into a deep mine.*

Note 2. We have

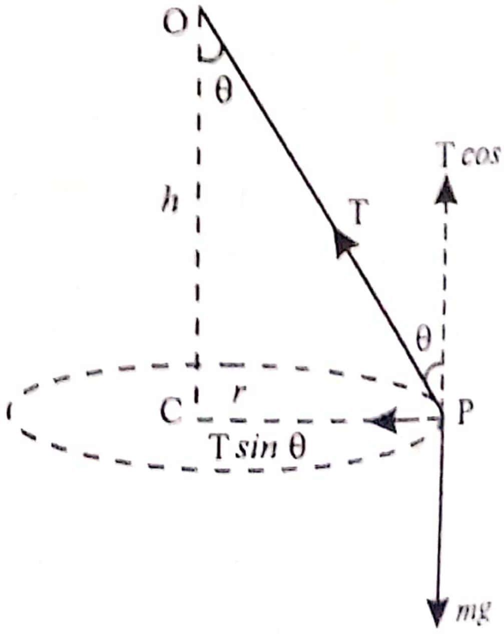
$$dn = -\frac{nh}{r} \text{ in } (i), \qquad dn = -\frac{nh}{2r} \text{ in } (ii)$$

$\therefore$ time lost by a clock when carried to a depth h is half of the time lost when it is carried to the same height h above the surface of earth.

12. 1481 metres 13. 296·2 meters

Art-6. Conical Pendulum

Definition. A heavy particle attached to one end of a light inextensible string, the other end of which is tied to a fixed point describes a horizontal circle with constant angular velocity so that the string itself describes a right circular cone with the fixed point as its vertex and the vertical line through the fixed point as the axis of the cone, the system thus formed is called a conical pendulum.



Let a heavy particle of mass m be attached to one end of a light inextensible string of length l and the other end of which is tied to a fixed point O. Let the particle describe a horizontal circle of radius r and centre C with constant angular velocity ω. Let P be the position of the particle at any time t and $\angle COP = \theta$.

The forces acting on the particle are :

(*i*) Its weight mg acting vertically downwards and

(*ii*) the tension T of the string along PO.

Since there is no motion in a vertical direction.

$\therefore$ $T \cos\theta = mg$ ...(1)

As the particle describes a horizontal circle of radius r with constant angular velocity ω, the centripetal force $m r \omega^2$ is supplied by the horizontal component $T \sin\theta$ of the tension T.

Now when we consider the horizontal circular motion of P, we get

$T \sin\theta = m r \omega^2$ ...(2)

But in Δ OCP, $\sin\theta = \frac{r}{l} \Rightarrow r = l \sin\theta$

$\therefore$ from (2), $T \sin\theta = m\, l \sin\theta \,.\, \omega^2$

But $\sin\theta = 0 \Rightarrow \theta = 0 \Rightarrow$ the string is vertical i.e. the system is no longer a conical pendulum.

$\therefore$ rejecting this, we get,

$$T = m\, l\, \omega^2$$ ...(3)

$\therefore$ from (1), we get,

$$m\, l\, \omega^2 \cos\theta = mg, \text{ or } \cos\theta = \frac{g}{l\omega^2}$$ ...(4)

We know that $\cos\theta \leq 1$

$\therefore$ motion in a conical pendulum is possible only when $\frac{g}{l\omega^2} \leq 1$

i.e. if $\omega^2 \geq \frac{g}{l}$ *i.e.* if $\omega \geq \sqrt{\frac{g}{l}}$

$\therefore$ *the least angular velocity required by a particle to move in a conical pendulum is* $\sqrt{\frac{g}{l}}$.

In Δ OCP, $\cos\theta = \frac{h}{l}$ or $\frac{g}{l\omega^2} = \frac{h}{l}$ [$\because$ of (4)]

$$\Rightarrow \quad h = \frac{g}{\omega^2}$$ ...(5)

which is independent of l

$\therefore$ *the depth of the particle below the fixed point is independent of the length of the string and is inversely proportional to the square of the angular velocity.*

Let T be the time of one revolution

$$\therefore \quad T = \frac{2\pi}{\omega} \Rightarrow T = \frac{2\pi}{\sqrt{g/h}} \qquad [\because \text{ of } (5)]$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{h}{g}}$$

$\therefore$ *the time of resolution varies directly as the square root of the depth of the particle below the fixed point.*

Let n be the number of revolutions made per second.

$$\therefore \quad \omega = 2\pi n \Rightarrow \frac{\omega}{2\pi} = n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{h}} \qquad [\because \text{ of } (5)]$$

$$\therefore \quad h = \frac{g}{4\pi^2 n^2}$$

From (3), $T = m\,l\,\omega^2 = m\,l\,(2\,n\,\pi)^2$

$$\therefore \quad T = 4\,\pi^2\,n^2 m\,l$$

Cor. 1. Since the tangent at P to the circle is perpendicular to each of the forces $m\,g$ and T.

$\therefore$ algebraic sum of the tangential components of the forces acting on the particle is zero.

$\therefore$ *speed of the particle is uniform.*

Cor. 2. Prove that the semi-vertical angle θ of a conical pendulum is given by $\tan\theta = \frac{v^2}{r\,g}$.

where v is the speed of the particle and r the radius of the horizontal circle described by the particle.

Proof. We have $T\cos\theta = m\,g$ ...(1)

and $T\sin\theta = m\,r\,\omega^2$

or $T\sin\theta = m\frac{v^2}{r}$ ...(2) $[\because \; v = r\,\omega]$

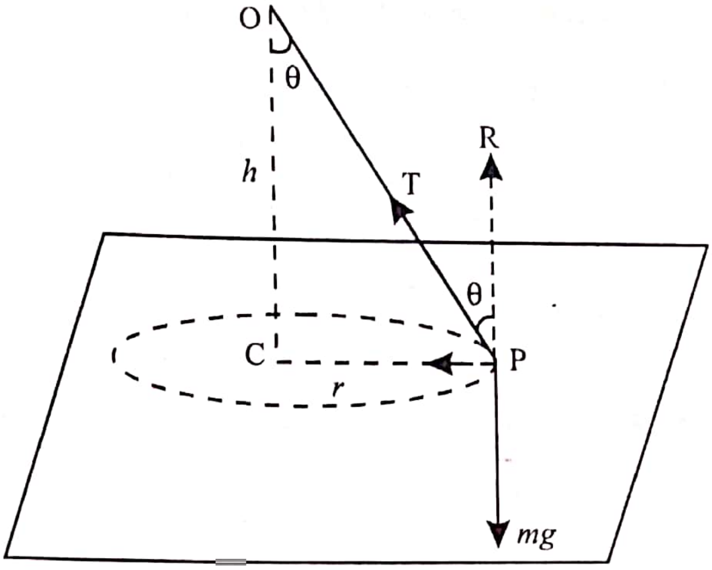
Dividing (2) by (1), we get,

$$\tan\theta = \frac{v^2}{r\,g}.$$

Art-7. If a particle moving in a conical pendulum is also constrained to move on a smooth horizontal surface at a depth h below the fixed end of the string, then show that the maximum angular velocity that the particle can acquire without losing contact with the surface is $\sqrt{\frac{g}{h}}$.

Proof : Since the particle moving in a conical pendulum is also constrained to move on a smooth horizontal surface.

$\therefore$ there is an additional force R, the normal reaction due to contact of the horizontal surface on the particle, acting vertically upward.



The equations of motion of the particle are

$$R + T\cos\theta = mg \quad ...(1)$$

and $T\sin\theta = m\,r\,\omega^2$

or $T\sin\theta = m\,l\sin\theta.\,\omega^2$

or $(T - m\,l\,\omega^2)\sin\theta = 0$

But $\sin\theta \neq 0$,

$$\therefore \quad T = m\,l\,\omega^2 \quad ...(2)$$

From (1) and (2), we get

$$R = mg - T\cos\theta = mg - m\,l\,\omega^2\cos\theta$$

$$= m\,g - m\,h\,\omega^2 \quad [\because h = l\cos\theta]$$

$$\therefore \quad R = m\,(g - h\,\omega^2)$$

Now the particle will remain in contact with the smooth horizontal surface if $R > 0$.

i.e. if $g > h\omega^2$ i.e. if $\omega^2 < \dfrac{g}{h}$ *i.e.* if $\omega < \sqrt{\dfrac{g}{h}}$

Hence the maximum angular velocity that the particle can acquire without losing contact with the surface is $\sqrt{\dfrac{g}{h}}$.