7

PROJECTILES

Art-1. Introduction

So far we have been dealing with motion of a particle along a straight line, the position of the particle being completely determined by a single measurement along the line from a given point on it. Now we consider the motion of the particle in a plane where the position of the particle P will be determined either by two distances x, y measured parallel to OX, OY in the plane or by r, the distance of the particle P from a fixed point O and the angle θ which OP makes with some fixed line OX in the plane.

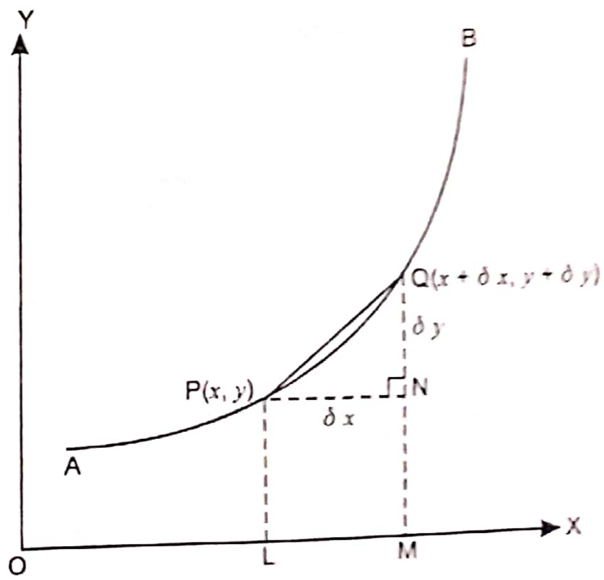
In the case of motion of a particle along a straight line, the direction is constant and so we consider only the magnitude of the velocities and accelerations with positive or negative signs, then resultant being obtained simply by their algebraic sum.

In the case of motion of a particle in a plane, we find the components of velocities and accelerations in two directions, the resultant is obtained by parallelogram law.

We already know that the rate of displacement of a moving particle is called its velocity and the rate of change of velocity of a moving particle is called its acceleration.

Art-2. Find the expressions for velocity and acceleration of a particle moving in a plane in rectangular co-ordinate system.

(P.U. 2010)



Proof. Let the particle be moving along the curve AB and let it move from the position P (x, y) at time t to the neighbouring point Q $(x + \delta x, y + \delta y)$ at time $t + \delta t$

The displacement PQ may be represented by its components PN and NQ parallel to the axes.

The displacement of the particle parallel to OX in time δt

$= PN = \delta x$

$\therefore$ resolved part of the velocity parallel to OX

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{\delta x}{\delta t} = \frac{dx}{dt}$$

Similarly the resolved part of the velocity parallel to OY

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

If V is the resultant velocity making an angle θ with OX, then

$$V \cos \theta = \frac{dx}{dt} \quad \ldots(1)$$

and $$V \sin \theta = \frac{dy}{dt} \quad \ldots(2)$$

Squaring and adding (1) and (2), we get,

$$V^2 (\cos^2 \theta + \sin^2 \theta) = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \quad \text{or} \quad V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$\therefore \quad V = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Dividing (2) by (1), we get,

$$\tan \theta = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \tan \phi,$$ where ϕ is the angle which the tangent to the curve at P makes with x-axis.

Hence the velocity at any point on the path acts along the tangent to the path at the point.

Now let u, v be the resolved part of the velocity of the particle parallel to the axes at time t and $u + \delta u$, $v + \delta v$ be the resolved parts of the velocity at time $t + \delta t$. Then acceleration along OX

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{\text{Change in velocity in time } \delta t \text{ along OX}}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{(u + \delta u) - u}{\delta t} = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta u}{\delta t} = \frac{du}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2 x}{dt^2}$$

and acceleration along OY

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{\text{Change in velocity in time } \delta t \text{ along OY}}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{(v + \delta v) - v}{\delta t} = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta v}{\delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2 y}{dt^2}$$

$\therefore$ resultant acceleration $= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$

and it acts at an angle $\tan^{-1}\left(\frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}\right)$ with OX.

Note : $\frac{d^2x}{dt^2} = u\frac{du}{dx}, \frac{d^2y}{dt^2} = v\frac{dv}{dy}$

Art-3. Equation of Motion of a Particle in a Plane

Let X and Y be the sum of resolved parts of the forces acting on the particle of mass m parallel to OX and OY respectively.

$\therefore$ the equations of motion are

$$m\frac{d^2x}{dt^2} = X \text{ and } m\frac{d^2y}{dt^2} = Y$$

Art-4. Projectiles

(*i*) **Projectile :** It is a body which is small enough to be regarded as a particle and which is projected in a direction oblique to the direction of gravity.

(*ii*) **Trajectory :** It is the path traced by the projectile.

(*iii*) **Velocity of Projection :** It is the velocity with which the particle is projected.

(*iv*) **Angle of Projection :** It is the angle which the direction of motion makes with the horizontal.

(*v*) **Range :** It is the distance measured between the point of projection and the point where the projectile hits a given plane through the point of projection.

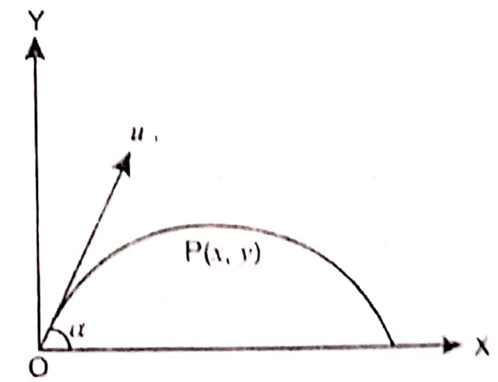
If the given plane is horizontal, then range is called **horizontal range.**

(*vi*) **Time of Flight :** It is the time taken to complete a particular range.

Art-5. A particle of mass m is projected from a fixed point with velocity u in a direction making an angle $\alpha \left(\neq \frac{\pi}{2}\right)$ with the horizontal. Neglecting air resistance, find its motion and show that its path is a parabola.

(P.U. 2011; G.N.D.U. 2012)

Proof. Let O, the point of projection, be taken as the origin and let the horizontal and the vertical lines through O be taken as the axes of x and y.



Let P (x, y) be the position of the particle after time t.

Resolved part of the acceleration at P along OX $= \frac{d^2x}{dt^2}$

Resolved part of the acceleration at P along OY $= \frac{d^2y}{dt^2}$

During the motion of the projectile, the only force acting on it is its weight mg acting vertically downwards.

$\therefore$ the equations of motion in the horizontal and vertical directions are

$$m\frac{d^2x}{dt^2} = 0 \text{ and } m\frac{d^2y}{dt^2} = -mg$$

or $$\frac{d^2x}{dt^2} = 0 \qquad ...(1)$$

and $$\frac{d^2y}{dt^2} = -g \qquad ...(2)$$

Integrating (1) and (2) $w.r.t$ t, we get,

$$\frac{dx}{dt} = c_1 \qquad ...(3)$$

and $$\frac{dy}{dt} = -gt + c_2 \qquad ...(4)$$

where c_1, c_2 are constants of integration.

Initially at O, when $t = 0$, $\frac{dx}{dt} = u\cos\alpha$, $\frac{dy}{dt} = u\sin\alpha$

$\therefore$ from (3) and (4), we get, $c_1 = u\cos\alpha$, $c_2 = u\sin\alpha$

$\therefore$ from (3), $$\frac{dx}{dt} = u\cos\alpha \qquad ...(5)$$

and from (4), $$\frac{dy}{dt} = u\sin\alpha - gt \qquad ...(6)$$

From (5), it is clear that the horizontal component of velocity will remain constant and equal to $u\cos\alpha$ throughout the motion.

Equations (5) and (6) give the components of velocity in the horizontal and vertical directions at any time t.

Integrating (5) and (6), $w.r.t$ t, we get,

$$x = u\cos\alpha\,.\,t + A \qquad ...(7)$$

and $$y = u\sin\alpha\cdot t - \frac{1}{2}gt^2 + B \qquad ...(8)$$

Initially at O, when $t = 0$, $x = 0$, $y = 0$

$\therefore$ from (7) and (8), we get, $A = 0$, $B = 0$

$\therefore$ from (7), $$x = u\cos\alpha\,.\,t \qquad ...(9)$$

and from (8), $$y = u\sin\alpha\,.\,t - \frac{1}{2}gt^2 \qquad ...(10)$$

Equations (9) and (10) given the position of the particle after time t.

Now to obtain the equation of path traced out by the particle, we eliminate t from (9) and (10).

From (9), $t = \dfrac{x}{u\cos\alpha}$

Putting this value of t in (10), we get,

$$y = u\sin\alpha\left(\frac{x}{u\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$

or $$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$$

which is the equation of the path of the projectile.

Now multiplying both sides by $-\dfrac{2u^2\cos^2\alpha}{g}$, we get,

$$-\frac{2u^2\cos^2\alpha}{g}y = -\frac{2u^2\cos\alpha\sin\alpha}{g}(x) + x^2$$

$$\therefore \quad x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x = -\frac{2u^2\cos^2\alpha}{g}y$$

Adding $\left(\dfrac{u^2\sin\alpha\cos\alpha}{g}\right)^2$ to both sides,

$$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} = -\frac{2u^2\cos^2\alpha}{g}y + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2}$$

or $$\left(x - \frac{u^2\sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right)$$

which is a parabola of the form $(x-h)^2 = -l(y-k)$

$\therefore$ the path of the particle is a parabola.

Art-6. **Find the latus rectum, the vertex, the focus, the height of the directrix of the parabola traced out by a projectile.**

(G.N.D.U. 2010, 2011; P.U. 2011)

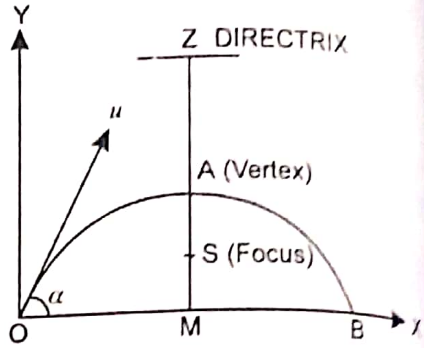
Proof. Let a particle of mass m be projected from a point O with velocity u in a direction making an angle α with the horizontal. Let the horizontal and vertical lines through O be taken as the axes of x and y respectively. Then the equation of the path of projectile is

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$$

or $\left(x - \dfrac{u^2 \sin\alpha\cos\alpha}{g}\right)^2 = -\dfrac{2u^2\cos^2\alpha}{g}\left(y - \dfrac{u^2\sin^2\alpha}{2g}\right)$

which is a parabola whose vertex is

$$\left(\frac{u^2 \sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$$



Latus rectum

$$= \frac{2u^2\cos^2\alpha}{g} = \frac{2}{g}(u\cos\alpha)^2$$

$$= \frac{2}{g}\text{ (horizontal component of velocity)}^2$$

Focus

Abscissa of focus S = OM = abscissa of vertex A

$$= \frac{u^2\sin\alpha\cos\alpha}{g}$$

Ordinate of focus S = MS = MA – SA

$$= \text{ordinate of vertex A} - \frac{1}{4}\text{ (latus rectum)}$$

$$= \frac{u^2\sin^2\alpha}{2g} - \frac{1}{4}\times\frac{2u^2\cos^2\alpha}{g} = -\frac{u^2}{2g}(\cos^2\alpha - \sin^2\alpha)$$

$$= -\frac{u^2\cos 2\alpha}{2g}$$

$\therefore$ focus S is $\left(\dfrac{u^2\sin\alpha\cos\alpha}{g}, -\dfrac{u^2\cos 2\alpha}{2g}\right)$

or $\left(\dfrac{u^2\sin 2\alpha}{2g}, -\dfrac{u^2\cos 2\alpha}{2g}\right)$

Height of the directrix

Height of the directrix = MZ = MA + AZ

$$= \text{ordinate of vertex A} + \frac{1}{4}\text{ (latus rectum)}$$

$$= \frac{u^2\sin^2\alpha}{2g} + \frac{1}{4}\times\frac{2u^2\cos^2\alpha}{g} = \frac{u^2}{2g}(\sin^2\alpha + \cos^2\alpha)$$

$$= \frac{u^2}{2g}$$

Note : From the height of the directrix, it is clear that all the trajectories which have the same point of projection and the same initial velocity have a common directrix.

Cor. Position of Focus S

Focus S is $\left(\dfrac{u^2 \sin 2\alpha}{2g}, -\dfrac{u^2 \cos 2\alpha}{2g}\right)$

S is above, on or below x-axis

as $-\dfrac{u^2 \cos 2\alpha}{2g} \gtreqless 0$

i.e., as $\cos 2\alpha \lesseqgtr 0$

i.e., as $2\alpha \lesseqgtr \dfrac{\pi}{2}$

i.e., as $\alpha \lesseqgtr \dfrac{\pi}{4}$

Art-7. A particle is projected with velocity u making an angle α with the horizontal. Find

(*i*) the time of flight

(*ii*) the horizontal range

(*iii*) the maximum horizontal range

(*iv*) the direction of projection for a given horizontal range

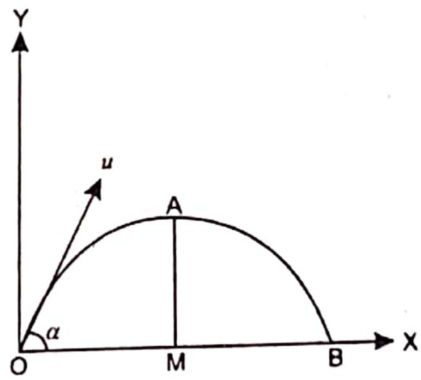
(P.U. 2011; G.N.D.U. 2014)

(G.N.D.U. 2012)

(*v*) the greatest height attained.

Proof. We know that the height of the projectile at time t is given by

$$y = u \sin \alpha \cdot t - \frac{1}{2} g t^2 \qquad \ldots(1)$$



(*i*) Now time of flight, T, is the time which the particle takes in reaching the horizontal plane through the point of projection.

Putting $y = 0$ in (1), we get,

$$0 = u \sin \alpha . t - \frac{1}{2} g t^2$$

or $$t\left(u \sin \alpha - \frac{1}{2} g t\right) = 0$$

$$\therefore \quad t = 0, \frac{2u \sin \alpha}{g}$$

But $t = 0$, corresponds the time when the particle is at the point of projection O

$$\therefore \quad T = \frac{2u \sin \alpha}{g}$$

(*ii*) Let R is the horizontal range OB

$\therefore$ R = the horizontal distance described by the particle in time of flight T

$$\therefore \quad R = (u \cos \alpha) \, . \, t = u \cos \alpha \, . \, \frac{2u \sin \alpha}{g}$$

$$= \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2}{g} \sin 2\alpha$$

(*iii*) The horizontal rang $R = \dfrac{u^2 \sin 2\alpha}{g}$ is maximum when $\sin 2\alpha$ is maximum i.e.

when $\sin 2\alpha = 1$

i.e., when $2\alpha = \dfrac{\pi}{2}$ *i.e.*, when $\alpha = \dfrac{\pi}{4}$

$\therefore$ the horizontal range is maximum when the angle of projection is $\dfrac{\pi}{4}$

and max. horizontal range $= \dfrac{u^2}{g}$.

(*iv*) Since $\dfrac{u^2}{g} \sin 2\left(\dfrac{\pi}{2} - \alpha\right) = \dfrac{u^2}{g} \sin (\pi - 2\alpha) = \dfrac{u^2 \sin 2\alpha}{g}$

Angles $\dfrac{\pi}{2} - \alpha$ and α give the same range

Hence with a given velocity of projection, to have a particular range, there are two directions of projection which are such that th inclination of the one with the horizontal is the same as that of the other with the vertical *i.e.*, they are equally inclined to the direction of the maximum range.

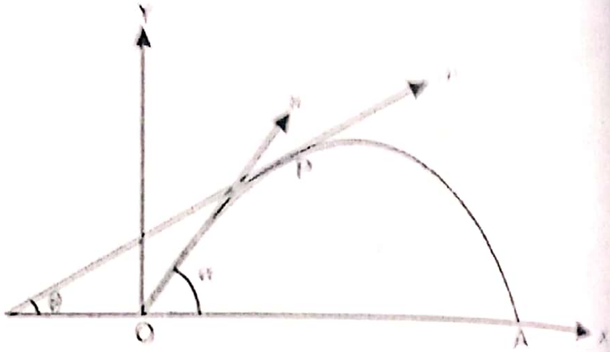
(G.N.D.U. 2014)

(*v*) Greatest height attained = MA = ordinate of vertex A

$$= \frac{u^2 \sin^2 \alpha}{2g}$$

Art-8. Find the velocity and direction of motion of a projectile after a given time t.

Proof. Let u be the velocity of projection and α the angle of projection. Let particle be at P after time t. Let v be its velocity at P (x, y) and θ the angle which the direction of velocity at P makes with horizontal. Therefore, we have,



$$x = u\cos\alpha \,.\, t \quad ...(1)$$

and $$y = u\sin\alpha \,.\, t - \frac{1}{2} g t^2 \quad ...(2)$$

Now $v\cos\theta$ = horizontal component of velocity after a time t

$$= \frac{dx}{dt} = u\cos\alpha \qquad [\because \text{ of } (1)]$$

$$\therefore \quad v\cos\theta = u\cos\alpha \quad ...(3)$$

and $v\sin\theta$ = vertical component of velocity after a time t

$$= \frac{dy}{dt} = u\sin\alpha - g\,t \qquad [\because \text{ of } (2)]$$

$$\therefore \quad v\sin\theta = u\sin\alpha - g\,t \quad ...(4)$$

Squaring and adding (3) and (4), we get,

$$v^2\cos^2\theta + v^2\sin^2\theta = u^2\cos^2\alpha + (u\sin\alpha - g\,t)^2$$

or $$v^2(\cos^2\theta + \sin^2\theta) = u^2(\cos^2\alpha + \sin^2\alpha) - 2\,u\sin\alpha \,.\, g\,t + g^2 t^2$$

or $$v^2 = u^2 - 2\,u\,g\sin\alpha \,.\, t + g^2 t^2$$

$$\therefore \quad v = \sqrt{u^2 - 2\,u\,g\sin\alpha \,.\, t + g^2 t^2}$$

which gives the magnitude of velocity at any time t.

Dividing (4) by (3), we get,

$$\tan\theta = \frac{u\sin\alpha - g\,t}{u\cos\alpha}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{u\sin\alpha - g\,t}{u\cos\alpha}\right)$$

which gives the direction of motion after time t.

Cor. Show that the magnitude of velocity at any point of a projectile is the same as would be acquired by a particle in falling freely a vertical distance from the level of the directrix to that point.

Sol. We have $v^2 = u^2 - 2\,u \sin\alpha \,.\, g\,t + g^2 t^2$

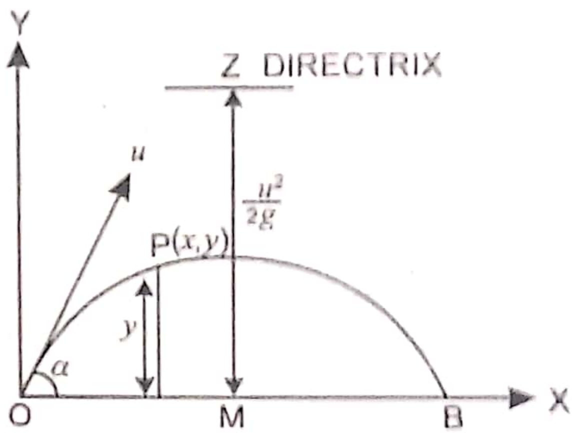
$$= u^2 - 2\,g\left(ut \sin\alpha - \frac{1}{2} g\,t^2\right)$$

$$= u^2 - 2\,g\,y = 2\,g\left(\frac{u^2}{2\,g} - y\right)$$

= 2 g (height of directrix – ordinate of P)

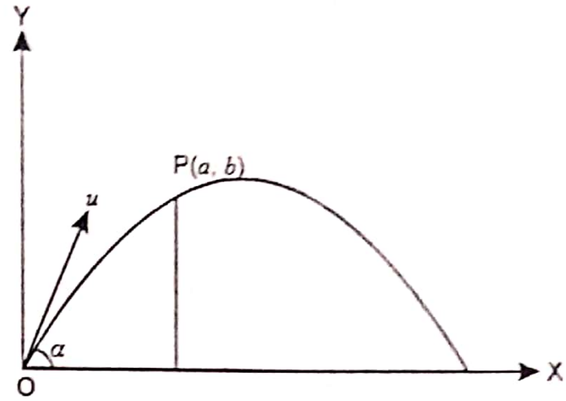
$\therefore\ v^2$ = 2 g (height of directrix above P)

Hence the result.



Art-9. Show that there are in general two directions of projection for a projectile to hit a given point with a given velocity of projection.

Also find the least velocity of projection to hit a given point.



Proof : Let u be the velocity of projection and α the angle of projection.

The equation of the trajectory is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad \ldots(1)$$

Let P(a, b) be given point. Then P(a, b) lies on (1) as projectile hits P.

$$\therefore \quad b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha}$$

or $$b = a\tan\alpha - \frac{ga^2}{2u^2}\sec^2\alpha$$

$$\Rightarrow \quad b = a\tan\alpha - \frac{ga^2}{2u^2}(1+\tan^2\alpha)$$

$$\Rightarrow \quad 2bu^2 = 2au^2\tan\alpha - ga^2 - ga^2\tan^2\alpha$$

$$\Rightarrow \quad ga^2\tan^2\alpha - 2u^2a\tan\alpha + (ga^2 + 2bu^2) = 0 \qquad \ldots(2)$$

This is a quadratic in $\tan\alpha$ giving us two values of $\tan\alpha$ and consequently two values of α, corresponding to each of which we get a direction of projection.

The hitting is possible when α is real *i.e.*, if the roots of (2) are real

i.e., if disc ≥ 0

i.e., if $(-2u^2a)^2 - 4.(ga^2)(ga^2 + 2bu^2) \geq 0$

i.e., if $u^4a^2 - g^2a^4 - 2ba^2gu^2 \geq 0$

i.e., if $u^4 - g^2a^2 - 2gbu^2 \geq 0$

i.e., if $u^4 - 2bgu^2 \geq a^2g^2$

i.e., if $u^4 - 2bgu^2 + b^2g^2 \geq a^2g^2 + b^2g^2$

i.e., if $(u^2 - bg)^2 \geq g^2(a^2+b^2)$

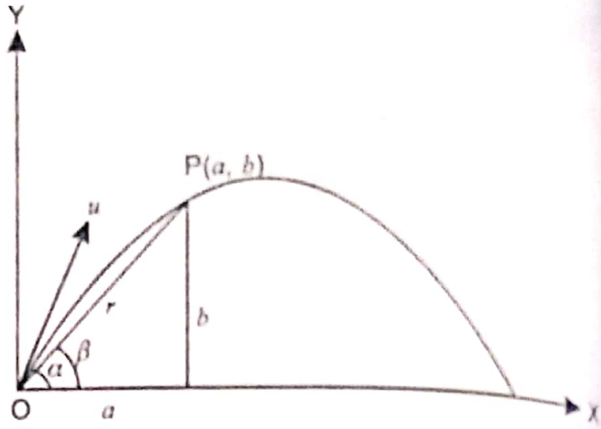
i.e., if $u^2 - bg \geq g\sqrt{a^2+b^2}$

i.e., if $u^2 \geq bg + g\sqrt{a^2+b^2}$

i.e., if $u^2 \geq g\left(b + \sqrt{a^2+b^2}\right)$

$\therefore$ least velocity of projection to hit (a, b) is $\sqrt{g\left(b+\sqrt{a^2+b^2}\right)}$

Cor. Let O be point of projection such that OP = r and OP makes an angle β with horizontal, then



$$r = \sqrt{a^2 + b^2}\ , \quad b = r \sin\beta$$

$\therefore$ least velocity of projection to hit (a, b) is

$$= \sqrt{g(r\sin\beta + r)}$$

$$= \sqrt{gr(1+\sin\beta)}$$

Note : The particle will not hit point, (a, b) if

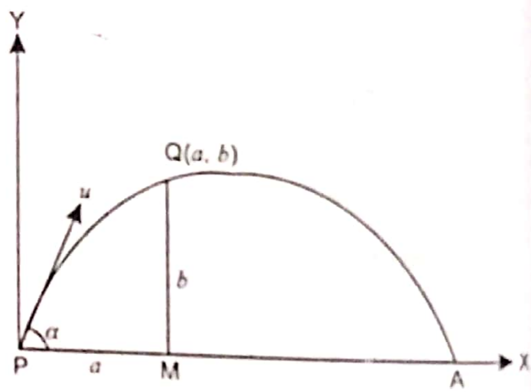
$$u^2 < g\left(b + \sqrt{a^2 + b^2}\right).$$

Art-10. Find the equation giving the two times corresponding to the two directions of projection for a projectile to hit a given point with a given velocity of projection. Show that product of times is

(*i*) independent of the initial velocity

(*ii*) equal to $\dfrac{2\,PQ}{g}$ where P is the point of projection and Q, the point to be hit.

Proof : Let u be the velocity of projection and α the angle of projection.



Let t be the time taken from the point of projection P to reach Q (a, b). Then

$$a = u\cos\alpha \,.\, t \qquad \ldots(1)$$

$$b = u\sin\alpha \,.\, t - \frac{1}{2}g t^2$$

or $$b + \frac{1}{2}g t^2 = u\sin\alpha \,.\, t \qquad \ldots(2)$$

Squaring and adding (1) and (2), we get,

$$a^2 + \left(b + \frac{1}{2}g t^2\right) = u^2 t^2 (\cos^2\alpha + \sin^2\alpha)$$

or $$a^2 + b^2 + \frac{1}{4}g^2 t^4 + b\,g\,t^2 = u^2 t^2$$

$$\therefore \quad g^2 t^4 + 4\,b\,g\,t^2 - 4\,u^2 t^2 + 4\,(a^2 + b^2) = 0$$

or $$g^2 t^4 + 4\,(b\,g - u^2)\,t^2 + 4\,(a^2 + b^2) = 0$$

$$\Rightarrow \quad t^4 + \frac{4}{g^2}(b\,g - u^2)\,t^2 + \frac{4}{g^2}(a^2 + b^2) = 0 \qquad \ldots(3)$$

It is a quadratic in t^2. Let t_1^2, t_2^2 be its roots.

$$\therefore \quad t_1^2 t_2^2 = \frac{4}{g^2}(a^2 + b^2)$$

$$\Rightarrow \quad t_1 t_2 = \frac{2}{g}\sqrt{a^2 + b^2}$$, which is independent of u, initial velocity.

Now P is (0, 0) and Q is (a, b)

$$\therefore \quad \text{PQ} = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

$$\therefore \quad t_1 t_2 = \frac{2}{g}\text{PQ} = \frac{2\,\text{PQ}}{g}.$$

ILLUSTRATIVE EXAMPLES

Art-11. Projectiles on an Inclined Plane

A particle is projected up an inclined plane of inclination β from a point O of the plane in the vertical plane through the line of greatest slope containing O, with initial velocity u at an elevation α to the horizon. Find

(*i*) the range on the plane

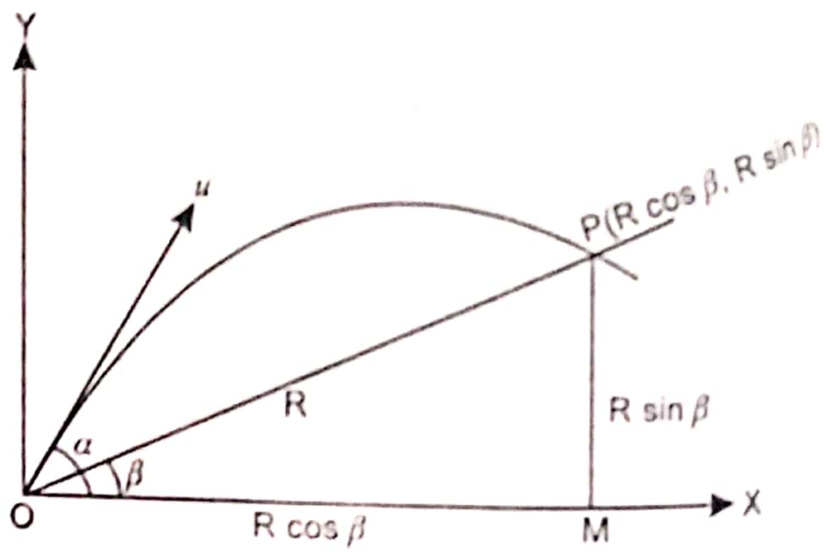
(*ii*) the maximum range for a given velocity of projection, u

(*iii*) the time of flight

Obtain corresponding results when the particle is projected down the plane.

(G.N.D.U. 2012)

Proof : Take O, the point of projection, as origin ; horizontal and vertical lines through O as axes. Let P be the point where the particle strikes the inclined plane. Let OP = R so that P is (R cos β, R sin β)



(*i*) The equation of the path of projectile is

$$y = x \tan \alpha - \frac{g x^2}{2 u^2 \cos^2 \alpha}$$

$\because$ P (R cos β, R sin β) lies on it

$$\therefore \quad R\sin\beta = R\cos\beta \,.\, \tan\alpha - \frac{g R^2 \cos^2\beta}{2u^2\cos^2\alpha}$$

$$\Rightarrow \quad \sin\beta = \cos\beta \,.\, \tan\alpha - \frac{g R\cos^2\beta}{2u^2\cos^2\alpha}$$

$$\Rightarrow \quad \frac{g R\cos^2\beta}{2u^2\cos^2\alpha} = \cos\beta \,.\, \frac{\sin\alpha}{\cos\alpha} - \sin\beta$$

$$\Rightarrow \quad \frac{g R\cos^2\beta}{2u^2\cos\alpha} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha}$$

$$\Rightarrow \quad \frac{g R\cos^2\beta}{2u^2\cos^2\alpha} = \frac{\sin(\alpha-\beta)}{\cos\alpha}$$

$$\Rightarrow \quad R = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

which gives the range on the plane.

(*ii*) Now $R = \dfrac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta} = \dfrac{u^2}{g\cos^2\beta}[2\sin(\alpha-\beta)\cos\alpha]$

$$= \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)+\sin(-\beta)]$$

$$\therefore \quad R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta]$$

Now u, g, β are constants

$\therefore$ R is maximum when $\sin(2\alpha-\beta)$ is maximum *i.e.* when $\sin(2\alpha-\beta)=1$ and this is so when $2\alpha-\beta = \dfrac{\pi}{2}$ or $\alpha = \dfrac{\pi}{4}+\dfrac{\beta}{2}$, which gives the angle of projection for maximum range up the plane.

$$\therefore \quad \text{max. range} = \frac{u^2}{g\cos^2\beta}(1-\sin\beta) = \frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)} = \frac{u^2}{g(1+\sin\beta)}$$

(*iii*) Let the particle strike the plane after time T

$\therefore$ OM = horizontal distance covered in time $T = u\cos\alpha \,.\, T$

$$\Rightarrow \quad R\cos\beta = u\cos\alpha \,.\, T \quad \Rightarrow T = \frac{R\cos\beta}{u\cos\alpha}$$

$$\Rightarrow \quad T = \frac{\cos\beta}{u\cos\alpha} \cdot \frac{2u^2 \sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

$$\Rightarrow \quad T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

When the particle is projected down the plane

Changing β to $-\beta$, we get

(*i*) Range down the inclined plane $= \dfrac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$

(*ii*) Maximum range down the inclined plane $= \dfrac{u^2}{g(1-\sin\beta)}$

(*iii*) Time of flight down the inclined plane $= \dfrac{2u\sin(\alpha+\beta)}{g\cos\beta}$

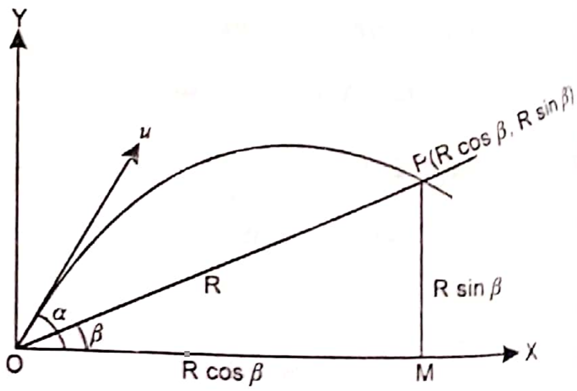
Cor. We know that R is maximum when

$$2\alpha - \beta = \frac{\pi}{2} \quad i.e.\ \alpha - \beta = \frac{\pi}{2} - \alpha$$

∴ *direction of projection of a projectile for maximum range up an inclined plane bisects the angle between the vertical through the point of projection and the line of greatest slope.* (G.N.D.U. 2013)

Art-12. A particle is projected with velocity u, making an angle α with the horizontal, up an inclined plane of inclination β to the horizon and strikes it at a point P. Find the velocity of the particle at P and the angle which this velocity makes with the horizontal.

Proof : Take O, the point of projection, as origin ; horizontal and vertical lines through O as axes. Let OP = R so that P is (R cos β, R sin β)



Let v be the velocities at P.

Now $v^2 = u^2 - 2gy$

$$\Rightarrow \quad v^2 = u^2 - 2g \cdot R\sin\beta$$

where $R = \dfrac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$

Also T = time of flight = time from O to P $= \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$

Let θ be the angle which the velocity at P makes with the horizontal.

$$\therefore \quad \tan\theta = \frac{u\sin\alpha - g\,T}{u\cos\alpha} = \tan\alpha - \frac{g}{u\cos\alpha}\,.\,T$$

$$= \tan\alpha - \frac{g}{u\cos\alpha}\cdot\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

$$= \tan\alpha - 2\left[\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}\right]$$

$$= \tan\alpha - 2\,[\tan\alpha - \tan\beta]$$

$$\therefore \quad \tan\theta = 2\tan\beta - \tan\alpha.$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan\beta - \tan\alpha)$$

ILLUSTRATIVE EXAMPLES