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RELATIVE MOTION

Art-1. Introduction

So far we have considered the motion of a particle w.r.t. a co-ordinate system which is not in motion. This reference frame is an inertial frame of reference. Now we will consider a co-ordinate system which itself is moving w.r.t. some other co-ordinate system. Measurement of displacement, velocity and acceleration made w.r.t. the moving co-ordinate system will be called *relative.*

The moving co-ordinate system can be one of the following three kinds:

(*i*) **Translation :** In this case, the co-ordinate system moves in such a way that its axes are parallel to corresponding axes of some reference frame.

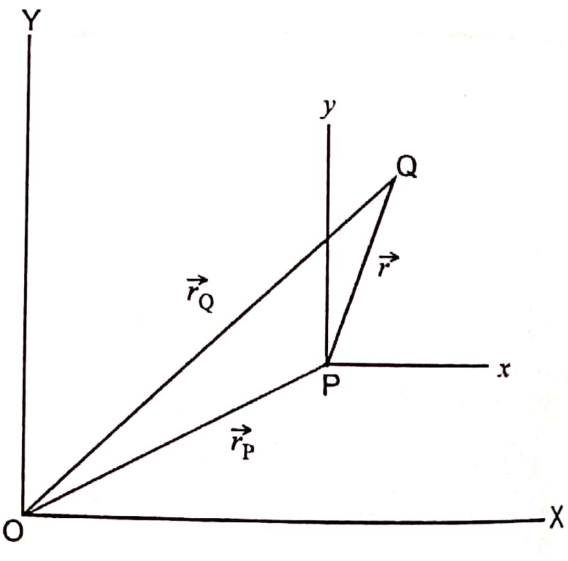
(*ii*) **Rotation :** In this case, the origin of reference in both the co-ordinate systems is the same, but the moving co-ordinate system rotates about the origin.

(*iii*) A combination of translation and rotation.

Art-2. Relative Displacement

Let OXY and P*xy* be two systems of co-ordinate axes.

Let $\vec{r}_P$ and $\vec{r}_Q$ be the position vectors of the two particles P and Q at time *t* with reference to co-ordinate system OXY. Then the vector $\overrightarrow{PQ} = \vec{r}$ is defined as the *relative displacement* (or the *apparent distance* of Q with respect to P).. The motion of Q is observed from the co-ordinate system P*xy*.



Now $\vec{r}_P + \vec{r} = \vec{r}_Q$

$\Rightarrow \quad \vec{r} = \vec{r}_Q - \vec{r}_P$

Let (x_P, y_P) and (x_Q, y_Q) be the co-ordinates of P and Q respectively w.r.t. co-ordinate system OXY. Let (*x*, *y*) be the co-ordinates of Q w.r.t. co-ordinate system P*xy*.

$\therefore \quad \vec{r} = \vec{r}_Q - \vec{r}_P \quad \Rightarrow \quad x = x_Q - x_P, \; y = y_Q - y_P$

$\therefore$ co-ordinate of Q relative of P are $(x_Q - x_P, \; y_Q - y_P)$

Since $\vec{r}_Q = \vec{r}_P + \vec{r}$

$\therefore$ displacement of Q in OXY = Relative displacement of Q + displacement of P.

Relative Velocity

We know that velocity of Q relative to P is defined as time rate of change of the relative displacement $\vec{r}$. Let $\vec{V}$ be the relative velocity of Q and $\vec{V}_P$, $\vec{V}_Q$ be the velocities of P, Q respectively.

Now $\vec{r} = \vec{r}_Q - \vec{r}_P \quad \Rightarrow \quad \dot{\vec{r}} = \dot{\vec{r}}_Q - \dot{\vec{r}}_P$

$$\Rightarrow \quad \vec{V} = \vec{V}_Q - \vec{V}_P$$

$\therefore$ *relative velocity of Q w.r.t. P is obtained by compounding the velocity of Q with the reversed velocity of P.*

Magnitude and direction of relative velocity of Q

Let the velocities $\vec{V}_P$, $\vec{V}_Q$ and $\vec{V}$ makes angles θ_1, θ_2 and θ with the positive direction of OX. Resolving these velocities along OX, OY, we get,

$$V \cos\theta = V_Q \cos\theta_2 - V_P \sin\theta_1 \quad \ldots(1)$$

and $$V \sin\theta = V_Q \sin\theta_2 - V_P \sin\theta_1 \quad \ldots(2)$$

Squaring and adding (1) and (2), we get,

$$V^2 (\cos^2\theta + \sin^2\theta) = V_Q^2 (\cos^2\theta_2 + \sin^2\theta_2) + V_P^2 (\cos^2\theta_1 + \sin^2\theta_1)$$
$$- 2 V_P V_Q (\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1)$$

$$\therefore \quad V^2 = V_P^2 + V_Q^2 - 2 V_P V_Q \cos(\theta_2 - \theta_1)$$

$$\therefore \quad V = \sqrt{V_P^2 + V_Q^2 - 2 V_P V_Q \cos(\theta_2 - \theta_1)}$$

which gives the magnitude of V.

Substituting the value of V in (1) and (2), we get the value of cos θ and sin θ and from where we get the value of θ i.e. the direction of the relative velocity.

Relative Acceleration

We know that the acceleration of Q relative to P is defined as the time rate of change of the relative velocity $\vec{V}$ of Q.

Now $\vec{V} = \vec{V}_Q - \vec{V}_P \quad \Rightarrow \quad \dot{\vec{V}} = \dot{\vec{V}}_Q - \dot{\vec{V}}_P$

$$\Rightarrow \quad \vec{f} = \vec{f}_Q - \vec{f}_P$$

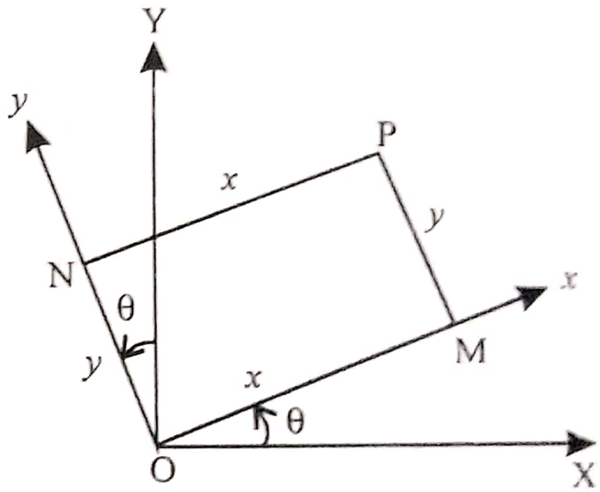
where $\vec{f}$ is the relative acceleration of Q and $\vec{f}_P$, $\vec{f}_Q$ are the acceleration of P, Q respectively.

$\therefore$ *relative acceleration of Q w.r.t. P is obtained by compounding the acceleration of Q with the reversed acceleration of P*

Magnitude and direction of relative acceleration of Q can be calculated like the magnitude and direction of relative velocity of Q.

Art-3. Motion Relative to Rotating Frame of Reference

Now we discuss the motion of a particle w.r.t. a frame of reference Oxy, which itself in rotating about a point O in a plane carrying a frame of reference OXY.



Let θ be the angular displacement of the rotating frame at time t, so that $\angle XOx = \theta$, and the angular velocity ω of the rotating frame is $\omega = \dfrac{d\theta}{dt}$ w.r.t. the fixed frame OXY.

Let P (x, y) be the moving point in the plane Oxy.

From P, draw PM ⊥ Ox and PN ⊥ Oy.

$$\angle Xoy = \frac{\pi}{2} + \theta \quad \Rightarrow \frac{d}{dt}(\angle XOy) = \frac{d\theta}{dt}$$

Now the radial and transverse velocities of M are $\frac{dx}{dt}$ along OM and $x\frac{d\theta}{dt}$ along MP and the radial and transverse velocities of N are $\frac{dy}{dt}$ along ON and $y\frac{d\theta}{dt}$ along PN produced.

∴ velocity of P parallel to Ox

= velocity of P ∥ Ox + velocity of P relative to N ∥ Ox

= velocity of N ∥ Ox + velocity of M along OM

$$= -y\frac{d\theta}{dt} + \frac{dx}{dt} = \frac{dx}{dt} - y\frac{d\theta}{dt}$$

Velocity of P parallel to Oy

= velocity of P ∥ Oy + velocity of P relative to M ∥ Oy

= velocity of M ∥ Oy + velocity of N along ON

$$= x\frac{d\theta}{dt} + \frac{dy}{dt}$$

Radial and transverse accelerations of M are

$$\frac{d^2x}{dt^2} - x\left(\frac{d\theta}{dt}\right)^2 \text{ along OM and } \frac{1}{x}\frac{d}{dt}\left(x^2\frac{d\theta}{dt}\right) \text{ along MP}$$

and radial and transverse accelerations of N are

$$\frac{d^2y}{dt^2} - y\left(\frac{d\theta}{dt}\right)^2 \text{ along ON and } \frac{1}{y}\frac{d}{dt}\left(y^2\frac{d\theta}{dt}\right) \text{ along PN produced.}$$

∴ acceleration of P parallel to Ox

= acceleration of P ∥ Ox + acceleration of P relative to N ∥ Ox

= acceleration of N ∥ Ox + acceleration of M along OM

$$= -\frac{1}{y}\frac{d}{dt}\left(y^2\frac{d\theta}{dt}\right) + \frac{d^2x}{dt^2} - x\left(\frac{d\theta}{dt}\right)^2$$

Acceleration of P parallel to Oy

= acceleration of P ∥ Oy + acceleration of P relative to M ∥ Oy

= acceleration of M ∥ Oy + acceleration of N along ON

$$= -\frac{1}{x}\frac{d}{dt}\left(x^2\frac{d\theta}{dt}\right) + \frac{d^2y}{dt^2} - y\left(\frac{d\theta}{dt}\right)^2$$