8

WORK, POWER AND ENERGY

Art-1. When a body moves due to the action of a force, the force is said to do work.

In order to distinguish it from various other kinds of work, such work is called Mechanical work.

(*a*) When the force is constant, the work done by it is equal to the product of the force and the displacement of its point of application in the direction of the force.

Let F be the force, s the displacement and W the work done. Then

(*i*) if the displacement takes place in the direction and line of action of the force, then

$$W = F s ;$$

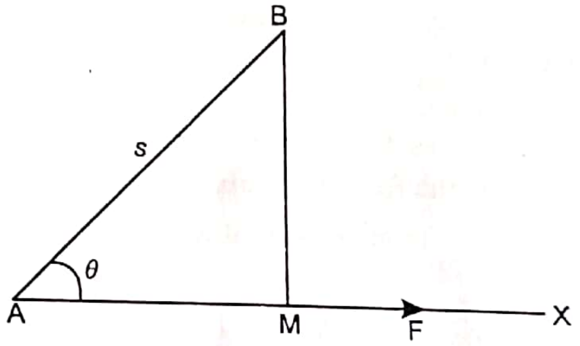
(*ii*) if the displacement is in the same line as the force but in the opposite direction, the work done by the force is negative

i.e. $W = -F s$;

(*iii*) if the displacement takes place along a line at an angle θ to the line of action of the force, then

W = F. (Projection of s in the line of action of the force)

$$= F s \cos\theta$$



In the figure force F acts along AX, the displacement is AB = s making an angle θ with AX and BM ⊥ AX so that

projection of displacement s (= AB) on AX is AM = $s \cos\theta$

Since $F \cdot s\cos\theta = s \cdot F\cos\theta$

∴ the work done by the force is equal to the product of displacement and the resolved part of the force in the direction of the displacement.

It is clear that W is positive or negative according as θ is acute or obtuse.

If $\theta = 90°$, $W = F s \cos 90° = 0$. Thus force does no work if the displacement takes place in a direction perpendicular to the line of action of the force.

In vector notation,

$$W = \vec{F} \cdot \vec{AB}$$

Illustration

Let W denote the work done by the force of gravity on a body of mass m.

(*i*) When the body falls down vertically through a height h,

$$W = m g h$$

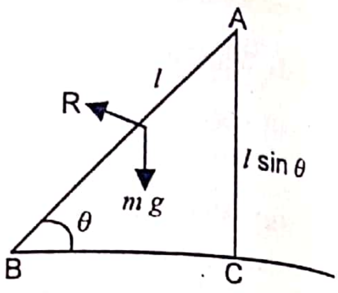
(*ii*) When the body rises vertically through a distance h.

$$W = -m g h$$

i.e. $m g h$ units of work is required in raising the body through the height h.

(*iii*) When a body slides down inclined plane of inclination θ through a length l of the plane, then

$$W = m g (l \sin \theta) = m g l \sin \theta$$



(*iv*) When the body moves up the plane, then

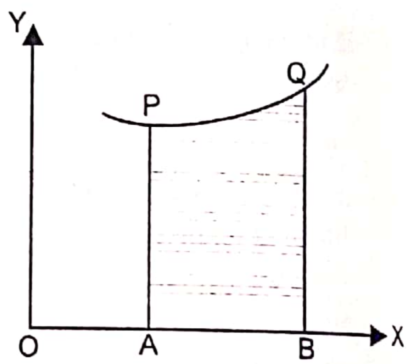
$$W = -m g l \sin \theta$$

(*v*) When $\theta = 0$, $W = 0$ $[\because \sin \theta = \sin 0 = 0]$

$\therefore$ if the body moves along a horizontal plane, the force of gravity does no work.

(*b*) When the force is variable and displacement takes place along its line of action.

Let OX be the line of action of the force and let the point of application move from A to B such that OA = a, OB = b.



Let F be the magnitude of the force at P such that OP = x.

When the point of application moves a small distance δx from P, the small amount of work δW done by the force F is approximately = F δx.

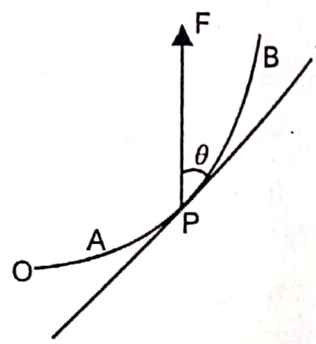
$\therefore$ total amount of work done is given by

$$W = \int_a^b F \, dx$$

Graphically, this work done is represented by the area ABQP.

(c) When the force is variable and the point of application moves along a plane curve.

Let a variable force move a particle from A to B along a given plane curve such that arc OA = s_1, arc OB = s_2 where O is a fixed point on the curve. Let P any point on the curve between A and B such that arc OP = s. Let Q be a point on the curve such that arc PQ = δs.



Let F be the force acting at P such that force F makes an angle θ with the direction of the tangent PT at P.

Since δs is small.

$\therefore$ direction of segment [PQ] coincides with that of the tangent at P.

$\therefore$ small amount δW of work done by F in taking the particle from P to Q is given by

$$\delta W = F\,\delta s \cos\theta = F_t\,\delta s$$

where F_t is the resolved part of F along the tangent to the curve at P.

$\therefore$ total work done by the force F in taking the particle from A to B is given by

$$W = \int_{s_1}^{s_2} F_t\,ds$$

Art-2. Units of Work

There are two types of units of work, absolute and gravitational (or practical).

(*a*) **The absolute unit of work** is the work done by an absolute unit of force in moving its point of application through a distance of unit length in the direction of the force.

In the English (F.P.S.) system, the absolute unit of work is the work done by a force of *one poundal* in moving its point of application through *one foot* in the direction of the force and is called a **Foot Poundal.**

In the French (C.G.S.) system, the absolute unit of work is the work done by a force of *one dyne* in moving its point of application through *one centimetre* in the direction of the force and is called an Erg.

In M.K.S. system, the absolute unit of work is a *Joule* and one joule is the work done by a force of one newton in moving its point of application through *one metre* in the direction of the force.

$$\begin{aligned} 1 \text{ Joule} &= 1 \text{ newton} \times 1 \text{ metre} \\ &= 10^5 \text{ dynes} \times 100 \text{ centimetres} \\ &= 10^7 \text{ ergs.} \end{aligned}$$

(*b*) **The gravitational (or practical) unit of work** is the work done in lifting the weight of a unit mass through a height of unit length.

In the English (F.P.S.) system, the gravitational unit of work is **Foot-Pound.** *It is work done in raising a mass of one pound vertically through a height one foot.*

$$\begin{aligned} 1 \text{ Foot-Pound} &= \text{Weight of mass of 1 pound} \times 1 \text{ foot} \\ &= 1\, g \text{ poundals} \times 1 \text{ foot} \\ &= g \text{ foot-poundals.} \end{aligned}$$

In the French (C.G.S.) system the gravitational unit of work is **Gramme-Centimetre.** *It is the work done in raising a mass of one gramme vertically through a height of one centimetre.*

$$\begin{aligned} 1 \text{ Gram-Centimetre} &= \text{Weight of mass of 1 gramme} \times 1 \text{ centimetre} \\ &= g \text{ dynes} \times 1 \text{ centimetre} \\ &= g \text{ ergs.} \end{aligned}$$

In the M.K.S. system, the gravitational unit of work is **kilogramme-metre**. *It is the work done in raising a mass of one kilogramme vertically through a height of one metre.*

1 Kilogramme-metre = Weight of a mass of 1 kilogramme × 1 metre

= 1000 g dynes × 100 centimetres

= 10^5 g ergs

= $\frac{g}{100}$ Joules.

Note : Since $g = 32$ ft/sec^2 or 981 cm/sec^2

∴ 1 Foot-pound = 32 poundals.

and 1 kilogramme-metre = 9·81 Joules.

Art-3. Work Done in Stretching an Elastic String

Prove that the work done in stretching an elastic string is equal to the product of the extension and mean of the initial and final tensions.

(P.U. 2011, G.N.D.U. 2012)

Proof. Let l be the natural length of the elastic string and λ the modulus of elasticity.

When the extension produced in the string is x, the tension T is given by

$$T = \lambda \frac{x}{l} \quad ...(1)$$

Since the string is being stretched, the force applied by an agent at any instant is always equal to the tension of the string at that instant.

∴ δW, the small amount of work done in stretching the string further distance δx against the tension T = T· δx.

$$\therefore \quad \delta W = \lambda \frac{x}{l}\, \delta x$$

∴ W, the total work done in stretching the string from a length l to a length $l + x$ is given by

$$W = \int_0^x \lambda \frac{x}{l}\, dx = \frac{\lambda}{l}\int_0^x x\, dx = \frac{\lambda}{l}\cdot\frac{x^2}{2}$$

$$\therefore \quad W = \frac{1}{2}\frac{\lambda}{l}x^2 \quad ...(1)$$

∴ *work done in stretching an elastic string varies as the square of the extension.*

Now the work done for an extension $x_1 = \frac{1}{2}\frac{\lambda}{l}x_1^2$

and the work done for an extension $x_2 = \frac{1}{2}\frac{\lambda}{l}x_2^2$

∴ work done in increasing the extension from x_1 to x_2

$$= \frac{1}{2}\frac{\lambda}{l}x_2^2 - \frac{1}{2}\frac{\lambda}{l}x_1^2 = \frac{\lambda}{2l}(x_2^2 - x_1^2) = \frac{\lambda}{2l}(x_2 - x_1)(x_2 + x_1)$$

$$= \left(\frac{\frac{\lambda x_1}{l} + \frac{\lambda x_2}{l}}{2}\right)(x_2 - x_1)$$

$$= \left(\frac{\text{initial tension} + \text{final tension}}{2}\right) \text{(extension)}$$

= (mean of initial and final tensions). (extension)

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$\therefore$ *the work done in stretching an elastic string is equal to the product of the mean of the initial and final tensions and the extension.*

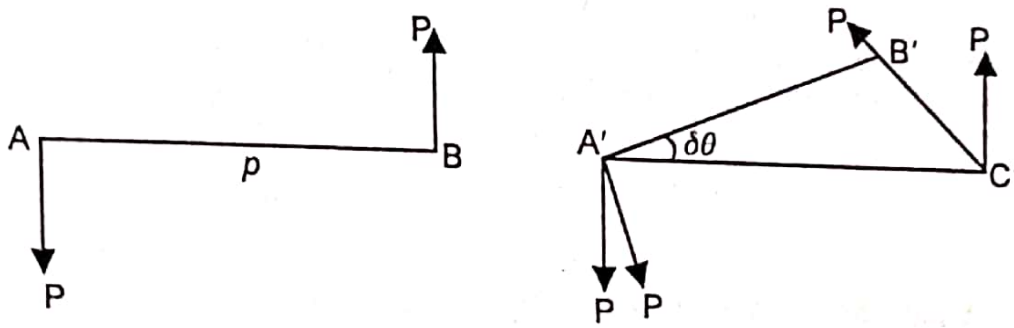
Art-4. Work Done by a Couple

Prove that

Work done by a couple = moment of the couple × circular measure of the angle of rotation.

Proof. Let M be the moment of the couple whose arm is AB = p and either of the forces is P.

$\therefore \quad M = P\,p$ ...(1)



Let A′ B′ be the new position of the arm AB such that ∠B′ A′ C′ = $\delta\theta$. Here A′C′ is parallel to AB.

Since the arm of the couple moves parallel to itself.

$\therefore$ forces forming the couple move parallel to themselves. Therefore no work is done as the two forces are equal in magnitude and opposite in direction.

Now force P acting A′ does no work as its point of application does not move

Work done by force at C′ = P × C′B′ = P × A′C′ $\delta\theta$

[$\because \sin\delta\theta = \delta\theta$ as $\delta\theta$ is small]

$= P\,p\,\delta\theta = M\,\delta\theta$ [$\because$ of (1)]

Let W be the total work done

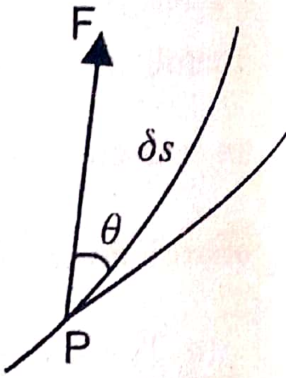
$$\therefore \quad W = \int_0^{\theta} M\,d\theta$$

= M · θ, where θ is the circular measure of angle of rotation

= Moment of couple × circular measure of the angle of rotation.

Art-5. Power

The rate of doing work of an agent is called its power. It is the amount of work which an agent can do in a unit time.



Let θ be the angle between the force F and the small displacement δs gone by its point of application.

$\therefore \quad \delta W = F\, \delta s \cos\theta$, where δW is small amount of work done.

$$\therefore \quad \frac{dW}{dt} = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta W}{\delta t} = \underset{\delta t \to 0}{\text{Lt}} \frac{(F\,\delta s \cos\theta)}{\delta t}$$

$$= F\cos\theta \underset{\delta t \to 0}{\text{Lt}} \frac{\delta s}{\delta t} = F\cos\theta \frac{ds}{dt} = F\,V\cos\theta$$

$= F \cdot V_F$ where V_F is the component of velocity in the direction of F.

Note 1. If $\theta = 0$, then $\dfrac{dW}{dt} = F\,V$

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$\therefore$ when force and displacement are in the same direction, then $\frac{dW}{dt} = FV$.

Note 2. Units of Power

In F.P.S. system, the unit of power is **Horse-Power** (H.P.). One Horse-Power is 550 foot-pounds work done in one second.

If F lbs, is the force applied and V ft/sec is the velocity of the particle, then the amount of work done in unit time is FV foot-pounds.

$$\therefore \quad \text{Horse-Power} = \frac{FV}{550}$$

In M.K.S. system, the unit of power is **Watt.** It is the rate of doing one Joule of work per second.

One Horse-Power = 746 Watts (nearly)

Relation between M.K.S. and F.P.S. units of power

In M.K.S. system, the unit of power is one Watt or Joule per second

$$\therefore \quad \text{H.P.} = \frac{\text{Work done per second in Joules}}{746} \qquad [\because 1 \text{ H.P.} = 746 \text{ watts}]$$

$$= \frac{\text{Force in newtons} \times \text{velocity in m/sec}}{746}$$

$$= \frac{\text{Force in kg. wt.} \times g \times \text{velocity in m/sec}}{746}$$

$$= \frac{\text{Force in kg. wt.} \times \text{velocity in m/sec}}{75}$$

$$\left[\because \frac{g}{746} = \frac{9 \cdot 81}{746} = \frac{1}{75} \text{ nearly}\right]$$

4. 2940 Joules/sec, 5390 Joules/sec   5. 420

Art-6. Conservative Forces

(G.N.D.U. 2010)

A system of forces, which is of such a nature that if a particle, after describing any path in the field of forces returns to its original position and the total work done during the displacement is zero, is called a conservative system of forces.

Let us take a body of mass m. Let it be taken to a height h. Then the work done by the gravity is $-mgh$. When the body falls back to the same level, under the action of gravity alone, work done is mgh. Therefore, total algebraic work done by gravity is zero. In general, if a body, after describing any path under the action of gravity returns to its original position, the total work done during the displacements it has gone, is zero. Thus gravity is conservative force.

Such is not the case with all the forces. If, for example, a body is dragged through a distances s against a constant frictional force F, the work done is Fs. To bring the body to its farmer position on the same oath an equal amount of work Fs is to be done again. Thus the total work performed to bring the body to its original position as 2 Fs and not zero. Forces of this type are called **non-conservative forces.**

A force field in which all the forces are conservative, is called a **conservative field.**

Note. In the case of conservative forces, the work done in taking a particle from one position to another depends only on the initial and final positions and is independent of the path followed by the particle.

Art-7. Energy

(G.N.D.U. 2010)

Energy of a body is its capacity to do work.

It is of two kinds : Kinetic Energy and Potential Energy.

(i) Kinetic Energy : *It is the capacity of the body to do work by virtue of its motion. It is measured by the amount of work done that the external forces will do to bring it to rest.*

For example, a bullet fired horizontally from a rifle possesses kinetic energy. If it pierces a vertical wall, the force of resistance offered by the wall will reduce the bullet to rest. The algebraic work done by the resistance is negative as it acts against the direction of the displacement. This work, with sign changed, is the kinetic energy of the bullet before piercing the wall.

(*ii*) Potential Energy : *It is the capacity of the body to do work by virtue of its position. It is measured by the amount of the work, the forces acting on it will do in bringing it from its present position to some standard position, usually called the zero position.*

For example, the potential energy of a stretched elastic string is the amount of work that its tension will do in making the string resume its unstretched form, the position of zero potential energy.

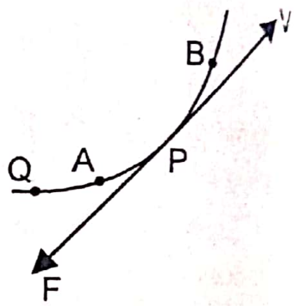
Note 1. The units of energy are same as those of work.

Note 2. Kinetic and Potential energies are written as K.E. and P.E.

Art-8. Expressions for Kinetic Energy

Show that the kinetic energy of a particle of mass m moving with a magnitude of velocity V is $\frac{1}{2} m V^2$. (G.N.D.U. 2010, 2011)

Proof : Let the particle of mass m move along a curve on which O is fixed point from which actual distances are measured. Let the force F, which acts in a direction opposite to the direction in which particle moves, reduce the velocity V of the particle at A to velocity zero at B such that arc OA = s_1 , arc OB = s_2. Let v be the velocity of particle at any point P such that arc OP = s.



∴ work done by force F in taking the particle from A to B

$$= \int_{s_1}^{s_2} (-F)\, ds = -\int_{s_1}^{s_2} F\, ds$$

$$= -\int_{s_1}^{s_2} m f\, ds = -m \int_{s_1}^{s_2} v \frac{dv}{ds} ds = -m \int_{V}^{0} v\, dv$$

$$= -m \left[\frac{v^2}{2}\right]_V^0 = -m\left[0 - \frac{V^2}{2}\right] = \frac{1}{2} m V^2$$

∴ kinetic energy of the particle = $\frac{1}{2} m V^2$

Cor. At P, the K.E. of the particle is $\frac{1}{2} m v^2$.

Now $\frac{d}{ds}\left(\frac{1}{2} m v^2\right) = \frac{1}{2} m \frac{d}{ds}(v^2)$

$$= \frac{1}{2} m \cdot 2v \frac{dv}{ds}$$

$$= m \times v \frac{dv}{ds}$$

= mass × acceleration

= impressed force in the direction of motion.

∴ *the rate of change of K.E. w.r.t. space is equal to the magnitude of the impressed force.*

(G.N.D.U. 2010)

Art-9. Potential Energy and Work

Show that the work done by the conservative forces in taking the system from a position P to another position Q equals the difference of potential energy at P and that at Q, both being taken with respect to an arbitrary standard position O.

Proof : Since the forces acting on the system are conservative.

∴ work done in taking the system from a position P to the position O is independent of the path followed and depends only upon the relative position P and O. Let V_P denote the unique value of the potential energy at P.

$$\therefore \quad V_P = \int_P^O F_t \, ds \qquad \ldots(1)$$

where F_t is the component of the force F along the tangent to the path of motion.

$$\text{Again} \quad V_Q = \int_Q^O F_t \, ds \qquad \ldots(2)$$

Subtracting (2) from (1), we get,

$$V_P - V_Q = \int_P^O F_t \, ds - \int_Q^O F_t \, ds = \int_P^Q F_t \, ds + \int_Q^O F_t \, ds - \int_Q^O F_t \, ds$$

$$\therefore \quad V_P - V_Q = \int_P^Q F_t \, ds$$

Hence the result.

Art-10. Work Done Against Gravity

Let a particle of mass m be at P such that OP = h where O is a point on the surface of earth.

$\therefore$ P.E. at P = $m\,g\,h$

Let particle be raised to Q such that OQ = h'

$\therefore$ P.E. at Q = $m\,g\,h'$

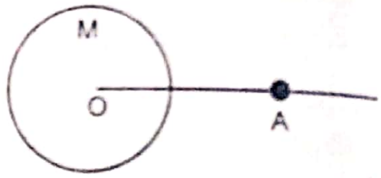
$\therefore$ P.E. in raising the particle from P to Q = $m\,g\,h - m\,g\,h'$

$= -\,m\,g\,(h' - h)$

Negative sign shows that work is done against gravity. It is assumed that h and h' are small as compared to the radius of the earth so that the force $m\,g$ due to gravity acting vertically downwards on the particle has the same value regardless of the position of the particle.

$\therefore$ work done by gravity when the particle falls from Q to P is $m\,g\,(h' - h)$. This is the P.E. of the particle at Q (OQ = h') relative to the position P (OP = h).

Art-11. Potential Energy of a Gravitational Field



Let a particle of mass m move from infinity to A in the gravitational field of mass M, where A is at a distance r from the centre O of gravitational field.

We know that the potential energy V of particle of mass m at A in the gravitational field of mass M is the work done against the field to move the mass m from infinity to A.

$$\therefore\ V = \int_{\infty}^{r} F\,dx = \int_{\infty}^{r} \frac{G\,M\,m}{x^2}\,dx = G\,M\,m \int_{\infty}^{r} x^{-2}\,dx$$

$$= G\,M\,m \left[\frac{x^{-1}}{-1}\right]_{\infty}^{r} = -\,G\,M\,m \left[\frac{1}{r}\right]_{\infty}^{r}$$

$$= -\,G\,M\,m \left[\frac{1}{r} - 0\right] = -\,\frac{G\,M\,m}{r}$$

Note : It is assumed that the gravitational P.E. is zero at infinity.

Art-12. Principle of Energy

If a particle moves under a system of conservative forces, then

(*i*) the change in kinetic energy produced is equal to the work done by the impressed forces. (G.N.D.U. 2013, 2015)

(This is known as **Principle of Work and Energy**)

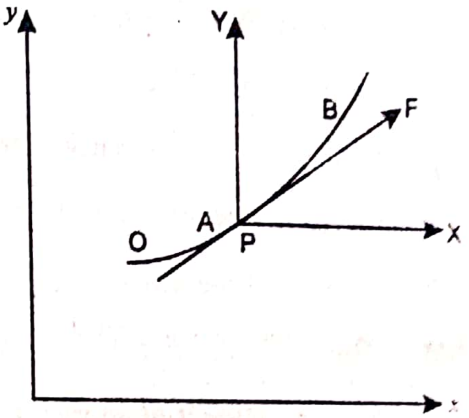
(G.N.D.U. 2007)

(*ii*) the sum of its kinetic and potential energies at any instant remains constant throughout the motion.

(This is known as **Principle of conservation of Energy**)

(P.U. 2013; G.N.D.U. 2012, 2013, 2014)

Proof. Let P (x, y) be the position of the particle of mass m at any time t where arc OA $= s$. Let ds be the displacement and F be the sum of components of the forces in the direction of ds.



$\therefore$ small work done by the force F is F ds.

Let X, Y be the sums of the components of the forces in the direction of the co-ordinate axes and dx, dy the displacements in these directions.

$\therefore$ small work done $= X\,dx + Y\,dy$

$\therefore$ total work done when the particle moves from a point A to another point B is

$$\int_A^B F\,ds = \int_A^B (X\,dx + Y\,dy) \qquad \ldots(1)$$

$$\text{Now} \quad \int_A^B F\,ds = \int_A^B m\,v\,\frac{dv}{ds}\,ds = m\int_{v_1}^{v_2} v\,dv = m\left[\frac{v^2}{2}\right]_{v_1}^{v_2}$$

Where v_1, v_2 are the velocities of the particle at A, B respectively.

$$= \frac{1}{2}m\,(v_2^2 - v_1^2) = \frac{1}{2}m\,v_2^2 - \frac{1}{2}m\,v_1^2$$

$$\therefore \quad \int_A^B F\,ds = K_B - K_A \qquad \ldots(2)$$

where K stands for the K.E.

$\Rightarrow$ work done by the impressed forces is equal to the change in kinetic energy

Since the system of forces is conservative.

$$\therefore \quad X = -\frac{\partial V}{\partial x},\; Y = -\frac{\partial V}{\partial y}, \text{ where V is the P.E.}$$

$$\therefore \quad \int_A^B (X\,dx + Y\,dy) = \int_A^B \left(-\frac{\partial V}{\partial x}\,dx - \frac{\partial V}{\partial y}\,dy\right) = -\int_A^B \left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy\right)$$

$$= -\int_A^B dV = -[V]_A^B = -[V_B - V_A]$$

$$\therefore \quad \int_A^B (X\,dx + Y\,dy) = V_A - V_B \qquad \ldots(3)$$

From (1), (2) and (3), we get,

$$K_B - K_A = V_A - V_B$$

$$\therefore \quad K_A + V_A = K_B + V_B$$

$\Rightarrow$ sum of K.E. and P.E. at A = sum of K.E. and P.E. at B

Since A and B are arbitrary points.

$\therefore$ sum of the kinetic and potential energies remains constant.

Note : The principle of, conservation of energy, may be stated as :

The total amount of energy in the universe is constant, energy cannot be created or destroyed although it may be converted into various forms.