14

FUNCTION OF A COMPLEX VARIABLE

Art-1. Exponential Function

Exponential function of the complex variable $z = x + iy$ is defined as

Exponential z or $\exp.(z) = e^z = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \ldots \infty$

Art-2. Prove that $e^{x+iy} = e^x(\cos y + i \sin y)$.

Proof : We have

$$e^z = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \ldots$$

$$\therefore \quad e^{x+iy} = 1 + (x+iy) + \frac{(x+iy)^2}{\lfloor 2} + \frac{(x+iy)^3}{\lfloor 3} + \ldots$$

Putting $x = 0$, we get,

$$e^{iy} = 1 + iy + \frac{(iy)^2}{\lfloor 2} + \frac{(iy)^3}{\lfloor 3} + \ldots$$

or $$e^{iy} = \left(1 - \frac{y^2}{\lfloor 2} + \frac{y^4}{\lfloor 4} + \ldots\right) + i\left(y - \frac{y^3}{\lfloor 3} + \frac{y^5}{\lfloor 5} - \ldots\right)$$

$$\therefore \quad e^{iy} = \cos y + i \sin y \qquad \ldots(1)$$

Now $$e^{x+iy} = e^x . e^{iy}$$

$$\therefore \quad e^{x+iy} = e^x(\cos y + i \sin y) \qquad [\because \text{ of } (1)]$$

Cor. $$e^{iy} = \cos y + i \sin y$$

Changing i to $-i$,

$$e^{-iy} = \cos y - i \sin y.$$

Art-3. Prove that $e^{z_1} . e^{z_2} = e^{z_1 + z_2}$.

Proof : Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$\therefore \quad e^{z_1} = e^{x_1}(\cos y_1 + i\sin y_1)$ and $e^{z_2} = e^{x_2}(\cos y_2 + i\sin y_2)$

$\therefore \quad e^{z_1}.e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1) \times e^{x_2}(\cos y_2 + i\sin y_2)$

$= e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)]$

$= e^{(x_1+x_2)+i(y_1+y_2)} = e^{(x_1+iy_1)+(x_2+iy_2)} = e^{z_1+z_2}$

$\therefore \quad e^{z_1}.e^{z_2} = e^{z_1+z_2}$

Cor. Generalising this result, we get,

$$e^{z_1}.e^{z_2}......e^{z_n} = e^{z_1+z_2+....+z_n}$$

Put $z_1 = z_2 = ... = z_n = z$

$\therefore \quad e^z.e^z...e^z = e^{z+z+....+z}$

$\therefore \quad (e^z)^n = e^{nz}$

Art-4. Periodicity of Exponential Function e^z

(Pbi. U. 2011)

Prove that e^z is a periodic function.

Proof : We have $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$

Now we know that $\sin y$ and $\cos y$ are periodic functions and period of each is 2π.

i.e., $\sin(y + 2n\pi) = \sin y$, $\cos(y + 2n\pi) = \cos y$, where $n \in N$.

$\therefore \quad e^z = e^x[\cos(y+2n\pi) + i\sin(y+2n\pi)] = e^x.e^{i(y+2n\pi)}$

$= e^x e^{iy} e^{i2n\pi} = e^{x+iy+2n\pi i}$

$\therefore \quad e^z = e^{z+2n\pi i}$

$\therefore \quad e^z$ remains unchanged when z is increased by any multiple of $2\pi i$.

$\therefore \quad e^z$ is a periodic function of period $2\pi i$.

Note : (1) $e^{i\pi} = \cos\pi + i\sin\pi = -1$

(2) $e^{2n\pi i} = \cos 2n\pi + i\sin 2n\pi = 1$

Art-5. Prove that $\exp z = 1$ if and only if $z = 2n\pi i$, n being an integer.

Proof : (*i*) Assume that

$e^z = 1, \quad \therefore \quad e^{x+iy} = 1 \Rightarrow e^x.e^{iy} = 1$

$\therefore \quad e^x(\cos y + i\sin y) = 1 + i0$

Equating real and imaginary parts, we get,

$e^x \cos y = 1$ ...(1)

and $e^x \sin y = 0$

Squaring and adding (1) and (2), we get,

$$e^{2x}(\cos^2 y + \sin^2 y) = 1 + 0$$

$\therefore$ $e^{2x} = 1 \Rightarrow e^{2x} = e^0 \Rightarrow 2x = 0 \Rightarrow x = 0$

$\therefore$ from (1) and (2), we get, $\cos y = 1, \sin y = 0$

$\therefore$ $y = 2n\pi$, n being an integer

$\therefore$ $z = 0 + 2in\pi \Rightarrow z = 2in\pi$

$\therefore$ $e^z = 1 \Rightarrow z = 2n\pi i$

(*ii*) Assume that $z = 2n\pi i$

$\therefore$ $e^z = e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1 + i0 = 1$

Cor. If z_1 and z_2 are two complex numbers and $\exp(z_1) = \exp(z_2)$, then prove that $z_1 = z_2 + 2n\pi i$ for some integer n.

Proof: $\exp(z_1) = \exp(z_2)$

$\Rightarrow$ $e^{z_1} = e^{z_2}$

$\Rightarrow$ $e^{z_1 - z_2} = 1$

$\Rightarrow$ $z_1 - z_2 = 2n\pi i$, n being an integer

$\Rightarrow$ $z_1 = z_2 + 2n\pi i$, n being an integer

ILLUSTRATIVE EXAMPLES

Example 1. Split up $e^{(6+5i)^2}$ into real and imaginary parts.

(Pbi. U. 2008, 20..)

Sol. $(6+5i)^2 = 36 + 25i^2 + 60i = 36 - 25 + 60i = 11 + 60i$

$\therefore$ $e^{(6+5i)^2} = e^{11+60i} = e^{11}(\cos 60 + i \sin 60)$

$\therefore$ $R\left[e^{(6+5i)^2}\right] = e^{11}\cos 60,\ I\left[e^{(6+5i)^2}\right] = e^{11} \sin 60.$

Example 2. For any integer k, evaluate $\sum_{k=1}^{n}\left[\exp.\left(\frac{2\pi i k}{n}\right)\right]^r$.

(G.N.D.U. 20..)

Sol. Let $S = \sum_{k=1}^{n}\left[\exp.\left(\frac{2\pi i k}{n}\right)\right]^r = \sum_{k=1}^{n}\left[\exp.\left(\frac{2\pi i r}{n}\right)\right]^k$

$$= \sum_{k=1}^{n}\alpha^k, \text{ where } \alpha = \exp.\left(\frac{2\pi i r}{n}\right)$$

Now two cases arise :

Case I. When $(r, n) = 1$

$\therefore$ $\frac{r}{n}$ is not an integer

$\Rightarrow$ $\alpha \neq 1$

$\therefore$ $S = \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^n$

$$= \frac{\alpha(1-\alpha^n)}{1-\alpha} \qquad \left[\because S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$= \frac{\alpha(1-1)}{1-\alpha} \qquad \left[\because \alpha^n = \left\{\exp.\left(\frac{2\pi i r}{n}\right)\right\}^n = \exp.(2\pi i r) = 1\right]$$

$$= 0$$

Case II. When $\frac{r}{n}$ is an integer, then $\alpha = 1$

$\therefore$ $S = 1 + 1 + 1 + \ldots$ to n terms.

$= n$

EXERCISE 14 (a)

1. Split up into real and imaginary parts :

(i) $e^{5+i\frac{\pi}{2}}$ (ii) $e^{(5+3i)^2}$ (iii) $e^{(x-iy)^2}$ (iv) e^{z^2}

(P.U. 2002)

2. Express in the form $A + iB$:

(i) $e^i + e^{-i}$ (ii) $\exp.\left(\frac{x-a+iy}{x+a+iy}\right)$

3. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, find the real and imaginary parts of

(i) $e^{z_1+z_2}$ (ii) $e^{z_1 z_2}$

4. (a) Prove that $e^{e^z} = e^{e^x \cos y} \text{cis}\,(e^x \sin y)$.

(b) Separate $e^{e^{2+i3}}$ into real and imaginary parts.

(P.U. 2002)

5. Prove that $\frac{1+\cos\alpha + i\sin\alpha}{1-\cos\alpha + i\sin\alpha} = \cot\frac{\alpha}{2} \cdot e^{-i\left(\frac{\pi}{2}-\alpha\right)}$.

6. Prove that

(i) $[\sin(\alpha+\theta) - e^{i\alpha}\sin\theta]^n = e^{-n\theta}\sin^n\alpha$

(ii) $\sin(\alpha+n\theta) - e^{i\alpha}\sin n\theta = e^{-in\theta}\,.\sin\alpha$

(iii) $[\sin(\alpha-\theta) + e^{\pm i\alpha}\sin\theta]^n = \sin^{n-1}\alpha\,[\sin(\alpha-n\theta) + e^{\pm i\alpha}\sin n\theta]$

(P.U. 2013)

7. If α, β are the roots of the equation $x^2+x+1=0$, show that for any number n,

$$\alpha\, e^{n\alpha} + \beta\, e^{n\beta} = e^{-\frac{n}{2}}\left(\cos\frac{\sqrt{3}}{2}n + \sqrt{3}\sin\frac{\sqrt{3}}{2}n\right).$$

(P.U. 2011, 2012)

8. For any integer r, evaluate $\sum_{k=1}^{n-1}\left[\exp.\left(\frac{2\pi i k}{n}\right)\right]^r$.

(G.N.D.U. 2008)

ANSWERS

1. (i) Real part = 0, imaginary part = e^5

(ii) Real part = $e^{16}\cos 30$, imaginary part = $e^{16}\sin 30$

(iii) Real part = $e^{x^2-y^2}\cos 2xy$, imaginary part = $-\,e^{x^2-y^2}\sin 2xy$

(iv) Real part = $e^{x^2-y^2}\cos 2xy$, imaginary part = $e^{x^2-y^2}\sin 2xy$

2. (i) $2\cos 1 + i\cdot 0$ (ii) $e^a\cos\beta + i e^a\sin\beta$

3. (i) real part $e^{x_1+x_2}\cos(y_1+y_2)$; imaginary = $e^{x_1+x_2}\sin(x_1 y_2 - x_2 y_1)$

(ii) Real part = $e^{x_1 x_2 - y_1 y_2}\cos(x_1 y_2 - x_2 y_1)$;

imaginary part = $e^{x_1 x_2 - y_2 y_2}\sin(x_1 y_2 - x_2 y_1)$

4. (b) Real part = $e^{\alpha}\cos\beta$, imaginary part = $e^{\alpha}\sin\beta$

7. -7 or $n-1$

Art-6. Trigonometrical (or Circular) Functions of Complex Quantities

For all real values of x, we have,

$$e^{ix} = \cos x + i\sin x \quad ...(1)$$

$$e^{-ix} = \cos x - i\sin x \quad ...(2)$$

Adding (1) and (2), $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

Subtracting (2) from (1), $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

These are called Euler's Exponential values of $\sin x$ and $\cos x$ where $x \in \mathbf{R}$.

If $z = x + iy$, then the trigonometric functions of z are defined as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}, \quad \cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \quad \operatorname{cosec} z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

Art-7. Expansions of cos z and sin z

Prove that

(*i*) $\cos z = 1 - \dfrac{z^2}{\lfloor 2} + \dfrac{z^4}{\lfloor 4} - \ldots$ (*ii*) $\sin z = z - \dfrac{z^3}{\lfloor 3} + \dfrac{z^5}{\lfloor 5} - \ldots$

where $z = x + iy$

Proof. We know that

$$e^{iz} = 1 + iz + \frac{(iz)^2}{\lfloor 2} + \frac{(iz)^3}{\lfloor 3} + \frac{(iz)^4}{\lfloor 4} + \ldots$$

$$= 1 + iz - \frac{z^2}{\lfloor 2} - \frac{iz^3}{\lfloor 3} + \frac{z^4}{\lfloor 4} + \ldots$$

$$\therefore \quad e^{iz} = \left(1 - \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} + \ldots\right) + i\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} + \ldots\right) \qquad \ldots(1)$$

Similarly $$e^{-iz} = \left(1 - \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} + \ldots\right) - i\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} + \ldots\right) \qquad \ldots(2)$$

Adding (1) and (2), we get,

$$e^{iz} + e^{-iz} = 2\left(1 - \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} + \ldots\right)$$

or $$\frac{e^{iz}+e^{-iz}}{2} = 1-\frac{z^2}{\underline{|2}}+\frac{z^4}{\underline{|4}}+\dots$$

$\therefore$ $$\cos z = 1-\frac{z^2}{\underline{|2}}+\frac{z^4}{\underline{|4}}+\dots$$

Subtracting (2) from (1), we get,

$$e^{iz}-e^{-iz} = 2i\left(z-\frac{z^3}{\underline{|3}}+\frac{z^5}{\underline{|5}}-\dots\right)$$

$\therefore$ $$\frac{e^{iz}-e^{-iz}}{2i} = z-\frac{z^3}{\underline{|3}}+\frac{z^5}{\underline{|5}}-\dots$$

$\therefore$ $$\sin z = z-\frac{z^3}{\underline{|3}}+\frac{z^5}{\underline{|5}}-\dots$$

Art-8. Euler's Theorem

$e^{i\theta} = \cos\theta + i\sin\theta$ whether θ is real or complex.

Proof. For all values of θ, real or complex,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \dots(1)$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \qquad \dots(2)$$

$\therefore$ $$\cos\theta + i\sin\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}+\frac{e^{i\theta}-e^{-i\theta}}{2} \qquad [\because \text{ of } (1), (2)]$$

$\therefore$ $$\cos\theta + i\sin\theta = \frac{2e^{i\theta}}{2}$$

$\therefore$ $$e^{i\theta} = \cos\theta + i\sin\theta.$$

Art-9. Prove that circular functions are periodic and find their period.

Proof : (*i*) We have

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz}+e^{-iz}}{2}$$

If n is any + ve integer, then

$$\sin(z+2n\pi) = \frac{e^{i(z+2n\pi)}-e^{-i(z+2n\pi)}}{2i}$$

$$= \frac{e^{iz}.e^{2n\pi i} - e^{-iz}.e^{-2n\pi i}}{2i}$$

$$= \frac{e^{iz} - e^{iz}}{2i} \qquad [\because e^{2n\pi i} = e^{-2n\pi i} = 1]$$

$\therefore \quad \sin(z + 2n\pi) = \sin z$

Also $\cos(z + 2n\pi) = \dfrac{e^{i(z+2n\pi)} + e^{-i(z+2n\pi)}}{2i}$

$$= \frac{e^{iz}.e^{2n\pi i} + e^{-iz}.e^{-2n\pi i}}{2}$$

$$= \frac{e^{iz} + e^{-iz}}{2} \qquad [\because e^{2n\pi i} = e^{-2n\pi i} = 1]$$

$\therefore$ sin z and cos z remain unchanged when z is increased by any multiple of 2π.

$\therefore$ sin z and cos z are periodic functions of period 2π.

(*ii*) Now $\qquad \tan z = \dfrac{e^{iz} - e^{-iz}}{i[e^{iz} + e^{-iz}]}$

If n is any + ve integer, then

$$\tan(z + n\pi) = \frac{e^{i(z+n\pi)} - e^{-i(z+n\pi)}}{i[e^{i(z+n\pi)} + e^{-i(z+n\pi)}]} = \frac{e^{iz}.e^{n\pi i} - e^{-iz}.e^{-n\pi i}}{i[e^{iz}.e^{n\pi i} + e^{-iz}.e^{-n\pi i}]}$$

$$= \frac{e^{iz}.e^{2n\pi i} - e^{-iz}}{i[e^{iz}.e^{2n\pi i} + e^{-iz}]} \qquad \left[\because e^{-n\pi i} = \frac{1}{e^{n\pi i}}\right]$$

$$= \frac{e^{iz} - e^{-iz}}{i[e^{iz} + e^{-iz}]} \qquad [\because e^{2n\pi i} = e^{-2n\pi i} = 1]$$

$$= \tan z$$

$\therefore$ tan z remains unchanged when z is increased by any multiple of π.

$\therefore$ tan z is a periodic function with period π.

As we proved in (*i*) and (*ii*), in the same way we can prove that sec z, cosec z are periodic functions with period 2π and cot z is periodic with π.

Hence all the circular functions of complex variable are periodic.

Art-10. Trigonometric Formulae for Circular Functions of Complex Variable

Prove that

(*i*) $\sin^2 z + \cos^2 z = 1$ $\qquad$ (*ii*) $\sin(-z) = -\sin z$

(*iii*) $\cos(-z) = \cos z$ (*iv*) $\tan(-z) = -\tan z$

(*v*) $\sin 2z = 2 \sin z \cos z$

(*vi*) $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

(*vii*) $\sin 3z = 3 \sin z - 4 \sin^3 z$ (*viii*) $\cos 3z = 4\cos^3 z - 3\cos z$

(*ix*) $\tan 3z = \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$

(*x*) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(*xi*) $\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$

(*xii*) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(*xiii*) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_2 \sin z_2$

(*xiv*) $\tan(z_1 + z_2) = \dfrac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$ (*xv*) $\tan(z_1 - z_2) = \dfrac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2}$

(*xvi*) $\sin z_1 + \sin z_2 = 2 \sin \dfrac{z_1 + z_2}{2} \cos \dfrac{z_1 - z_2}{2}$

(*xvii*) $\sin z_1 - \sin z_2 = 2 \cos \dfrac{z_1 + z_2}{2} \sin \dfrac{z_1 - z_2}{2}$

(*xviii*) $\cos z_1 + \cos z_2 = 2 \cos \dfrac{z_1 + z_2}{2} \cos \dfrac{z_1 - z_2}{2}$

(*xix*) $\cos z_1 - \cos z_2 = -2 \sin \dfrac{z_1 + z_2}{2} \sin \dfrac{z_1 - z_2}{2}$

Proof. (*i*) L.H.S. $= \sin^2 z + \cos^2 z = \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2$

$$= -\frac{1}{4}[e^{2iz} + e^{-2iz} - 2] + \frac{1}{4}[e^{2iz} + e^{-2iz} + 2]$$

$$= -\frac{1}{4}e^{2iz} - \frac{1}{4}e^{-2iz} + \frac{1}{2} + \frac{1}{4}e^{2iz} + \frac{1}{4}e^{-2iz} + \frac{1}{2} = 1$$

= R.H.S.

(*x*) We have

$$e^{i(z_1 + z_2)} = e^{iz_1} . e^{iz_2}$$

$\therefore$ $\cos(z_1 + z_2) + i \sin(z_1 + z_2) = (\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2)$

$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i(\cos z_1 \sin z_2 + \sin z_1 \cos z_2)$

Equating imaginary parts,

$\sin(z_1 + z_2) = \cos z_1 \sin z_2 + \sin z_1 \cos z_2$

$\therefore$ $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

Note : Students can prove the remaining parts themselves.

ILLUSTRATIVE EXAMPLES

Example 1. Prove that for all $z \in \mathbf{C}$, $\sin z = \dfrac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}}$

Sol. R.H.S. $= \dfrac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}} = \dfrac{2\dfrac{e^{i\frac{z}{2}}-e^{-i\frac{z}{2}}}{i\left(e^{i\frac{z}{2}}+e^{-i\frac{z}{2}}\right)}}{1+\left[\dfrac{e^{i\frac{z}{2}}-e^{-i\frac{z}{2}}}{i\left(e^{i\frac{z}{2}}+e^{-i\frac{z}{2}}\right)}\right]^2}$

$$= \frac{2i\left(e^{i\frac{z}{2}}-e^{-i\frac{z}{2}}\right)\left(e^{i\frac{z}{2}}+e^{-i\frac{z}{2}}\right)}{i^2\left(e^{i\frac{z}{2}}+e^{-i\frac{z}{2}}\right)^2+\left(e^{i\frac{z}{2}}-e^{-i\frac{z}{2}}\right)^2}$$

$$= \frac{2i(e^{iz}-e^{-iz})}{-\left(e^{i\frac{z}{2}}+e^{-i\frac{z}{2}}\right)^2+\left(e^{i\frac{z}{2}}-e^{-i\frac{z}{2}}\right)^2} = \frac{2i(e^{iz}-e^{-iz})}{-4e^{i\frac{\pi}{2}}.e^{-i\frac{\pi}{2}}}$$

$$= \frac{2i(e^{iz}-e^{-iz})}{-4}$$

$$= \frac{i^2(e^{iz}-e^{-iz})}{-2i} = \frac{e^{iz}-e^{-iz}}{2i} = \sin z$$

$= \text{L.H.S.}$

$$\therefore \quad \sin z = \frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}}$$

EXERCISE 14 (b)

1. If z is any complex number, then $\sec^2 z - \tan^2 z = 1$.

2. Prove that for all $\alpha, \beta \in C$, $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$.

3. Show that

   (i) $\cos(\alpha + i\beta) = \frac{1}{2}(e^{-\beta} + e^{\beta})\cos\alpha + i.\frac{1}{2}(e^{-\beta} - e^{\beta})\sin\alpha$

   (ii) $\sin(\alpha + i\beta) = \frac{1}{2}(e^{-\beta} + e^{\beta})\sin\alpha - i.\frac{1}{2}(e^{-\beta} - e^{\beta})\cos\alpha$

   (iii) $\cos(\alpha + i\beta) + i\sin(\alpha + i\beta) = e^{-\beta}(\cos\alpha + i\sin\alpha)$

(P.U. 2011)

Art-11. Logarithmic Functions

If $w = e^z$, where z and w are complex numbers, then z is called a logarithmic of w to the base e, and is written as $z = \log_e w$.

Art-12. Prove that $\log_e w$ is a many-valued function.

(G.N.D.U. 2007)

Proof. Let $e^z = w$

$\therefore \quad e^{z+2n\pi i} = e^z . e^{2n\pi i} = e^z . (\cos 2n\pi + i\sin 2n\pi) = e^z(1 + i.0) = e^z$

$\therefore \quad e^{z+2n\pi i} = w$

$\therefore$ by definition, $\log_e w = z + 2n\pi i$, where $n \in I$

$\therefore$ if z is a logarithm of w, so is $z + 2n\pi i$

$\therefore$ *logarithm of a complex number has infinite values and thus is a many-valued function.*

Note. The value $z + 2n\pi i$ is called the **general value** of $\log_e w$ and is denoted by $\text{Log}_e w$.

Thus $\text{Log}_e w = z + 2n\pi i$

$\therefore \quad \text{Log}_e w = 2n\pi i + \log_e w$

If we put $n = 0$ in the general value, we get the principal value of z *i.e.*, $\log_e w$.

$\therefore$ to find the general value of the logarithm of any complex number, add $2n\pi i$ to the principal value of the logarithm of that number.

Art-13. Prove that

(i) $\log N = x + 2n\pi i$ where $e^x = N$ such that N is a + ve real number and x is also real.

(ii) $\log(-N) = \pi i + \log N$, where N is positive.

(iii) $\log(i\beta) = \log\beta + \frac{\pi}{2} i$

Proof. (i) Here $e^x = N$

$\therefore \quad e^{x+2n\pi i} = e^x \cdot e^{2n\pi i} = e^x(\cos 2n\pi + i\sin 2n\pi) = e^x(1 + i\,0) = e^x$

$\therefore \quad e^{x+2n\pi i} = N$

$\therefore$ by definition, $\log N = x + 2n\pi i$.

Note. $\therefore$ one value of the logarithm of a + ve real number is real (when $n = 0$) and there are infinite number of imaginary values.

(ii) $-N = N(-1) = N(\cos\pi + i\sin\pi)$

$\therefore \quad -N = N e^{\pi i}$

$\therefore \quad \log(-N) = \log(N e^{\pi i}) = \log N + \log e^{\pi i} = \log N + \pi i \log e$

$\therefore \quad \log(-N) = \log N + \pi i$

Note. $\text{Log}(-N) = 2n\pi i + \log(-N)$

$\therefore \quad \text{Log}(-N) = 2n\pi i + \log(-N)$

Putting $n = 0$, the principal value of Log (– N) is log N

(iii) $$i\beta = \beta\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$\therefore \quad i\beta = \beta e^{\frac{\pi}{2}i}$$

$$\therefore \quad \log(i\beta) = \log\left(\beta e^{\frac{\pi}{2}i}\right) = \log\beta + \log e^{\frac{\pi}{2}i} = \log\beta + \frac{\pi}{2} i \log e$$

$$\therefore \quad \log(i\beta) = \log\beta + \frac{\pi}{2} i$$

Cor. Put $\beta = 1$

$$\therefore \quad \log i = \log 1 + \frac{\pi}{2} i$$

$$\therefore \quad \log i = \frac{\pi}{2} i \qquad [\because \log 1 = 0]$$

Note. Laws of Logarithms

These laws are same as for real numbers.

Art-14. Separate Log $(\alpha + i\beta)$ into real and imaginary parts.

(H.P.U. 2009, 2010)

Proof : Let $\alpha + i\beta = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts,

$$r\cos\theta = \alpha$$

$$r\sin\theta = \beta$$

Squaring and adding (1) and (2), we get,

$$r^2(\cos^2\theta + \sin^2\theta) = \alpha^2 + \beta^2$$

$$\therefore \quad r^2 = \alpha^2 + \beta^2$$

$$\therefore \quad r = \sqrt{\alpha^2 + \beta^2}$$

Dividing (2) by (1), $\tan\theta = \dfrac{\beta}{\alpha}$

$$\therefore \quad \theta = \tan^{-1}\frac{\beta}{\alpha}$$

$$\text{Log}(\alpha + i\beta) = 2n\pi i + \log(\alpha + i\beta) = 2n\pi i + \log[r(\cos\theta + i\sin\theta)]$$

$$= 2n\pi i + \log(re^{i\theta})$$

$$= 2n\pi i + \log r + \log e^{i\theta} = 2n\pi i + \log r + i\theta$$

$$= 2n\pi i + \log\left(\sqrt{\alpha^2+\beta^2}\right) + i\tan^{-1}\frac{\beta}{\alpha} \qquad [\because \text{ of (3) and (4)}]$$

$$\therefore \quad \text{Log}(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$

$$\therefore \quad R[\text{Log}(\alpha + i\beta)] = \frac{1}{2}\log(\alpha^2+\beta^2)$$

and $\quad I[\text{Log}(\alpha + i\beta)] = 2n\pi + \tan^{-1}\dfrac{\beta}{\alpha}$

Note. Putting $n = 0$, the principal value is given by

$$\log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}.$$

(G.N.D.U. 20..)

ILLUSTRATIVE EXAMPLES

Example 1. Express $\text{Log}[\text{Log}(\cos\theta + i\sin\theta)]$ in the form $A + iB$.

Sol. $\because \quad \log(\cos\theta + i\sin\theta) = \log e^{i\theta} = i\theta$

$$\therefore \quad \text{Log}(\cos\theta + i\sin\theta) = 2n\pi i + i\theta = (2n\pi + \theta)i$$

$$= (2n\pi + \theta)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$\therefore \quad \text{Log}(\cos\theta + i\sin\theta) = (2n\pi + \theta)\, e^{i\frac{\pi}{2}}$

$\therefore \quad \text{Log}[\text{Log}(\cos\theta + i\sin\theta)] = \text{Log}\left[(2n\pi + \theta)\, e^{i\frac{\pi}{2}}\right]$

$$= \text{Log}(2n\pi + \theta) + \log e^{i\frac{\pi}{2}}$$

$$= 2m\pi i + \log(2n\pi + \theta) + i\frac{\pi}{2}$$

$\therefore \quad \text{Log}[\text{Log}(\cos\theta + i\sin\theta)] = \log(2n\pi + \theta) + i(4m + 1)\frac{\pi}{2}$ where $m \in R$

Example 2. Find the general and principal value of $\log(-1 + i) - \log(-1 - i)$.

(G.N.D.U. 2003, 2007; Pbi. U. 2010)

Sol. Let $-1 + i = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts,

$$r\cos\theta = -1 \qquad ...(1)$$

$$r\sin\theta = 1 \qquad ...(2)$$

Squaring and adding (1) and (2)

$r^2 = 2, \quad \therefore \quad r = \sqrt{2}$

From (1) and (2), $\cos\theta = \dfrac{-1}{\sqrt{2}}$, $\sin\theta = \dfrac{1}{\sqrt{2}}$

Both these equations are satisfied when $\theta = \dfrac{3\pi}{4}$

$\therefore \quad -1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\, e^{i\frac{3\pi}{4}}$

$\therefore \quad \log(-1 + i) = \log\left[\sqrt{2}\, e^{i\frac{3\pi}{4}}\right] = \log 2^{\frac{1}{2}} + \log e^{i\frac{3\pi}{4}}$

$\therefore \quad \log(-1 + i) = \dfrac{1}{2}\log 2 + \dfrac{3\pi}{4} i \qquad ...(3)$

Changing i to $-i$,

$\log(-1 - i) = \dfrac{1}{2}\log 2 - \dfrac{3\pi}{4} i \qquad ...(4)$

Subtracting (4) from (3), we get,

$$\log(-1 + i) - \log(-1 - i) = \frac{3\pi}{2} i$$

Again $\text{Log}(-1 - i) = 2n\pi i + \log(-1 - i)$

$$\therefore \quad \text{Log}(-1-i) = 2n\pi i + \frac{1}{2}\log 2 - \frac{3\pi}{4}i$$

$$\text{Also} \quad \text{Log}(-1+i) = 2m\pi i + \frac{1}{2}\log 2 + \frac{3\pi}{4}i$$

$$\therefore \quad \text{Log}(-1+i) - \text{Log}(-1-i) = 2m\pi i - 2n\pi i + \frac{3\pi}{2}i = \left[2m - 2n + \frac{\pi}{2}\right]\pi$$

Example 3. Show that $i\log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x.$

(Pbi. U. 2010, 20...)

Sol. Let $x + i = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts,

$$r\cos\theta = x$$

$$r\sin\theta = 1$$

Dividing, $\tan\theta = \frac{1}{x}$ or $\cot\theta = x$

$$\therefore \quad \theta = \cot^{-1}x$$

$$\text{or} \quad \theta = \frac{\pi}{2} - \tan^{-1}x$$

$$\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$\text{L.H.S.} = i\log\left(\frac{x-i}{x+i}\right) = i\log\left[\frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)}\right] = i\log\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)$$

$$= i\log e^{-2i\theta} = i(-2i\theta) = 2\theta = 2\left(\frac{\pi}{2} - \tan^{-1}x\right) \quad [\because \text{of } (1)]$$

$$= \pi - 2\tan^{-1}x = \text{R.H.S.}$$

EXERCISE 14 (c)

1. Prove that $\text{Log}(-4) = 2\log 2 + (2n+1)\pi i.$ (H.P.U. 200...)

2. Find the general value of

   (i) $\log i$ (H.P.U. 2009) (ii) $\log(-i)$ (Pbi. U. 2009, 2010) (iii) $\log(-3)$ (iv) $\log(-5)$

3. Prove that $\sin(\log i^i) = -1.$

Art-15. The General Exponential Function

The general exponential function a^z is defined by the equation $a^z = e^{z \, \text{Log} \, a}$ where a and z are complex numbers.

Note. Since $\text{Log}\, a = 2n\pi i + \log a$

$\therefore$ a^z is a many-valued function and its principal value is obtained by putting $n = 0$.

Art-16. Express $(x + iy)^{\alpha + i\beta}$ in the form $A + iB$.

(H.P.U. 2007)

Proof : Let $x + iy = r(\cos\theta + i\sin\theta)$

so that $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Now $(x + iy)^{\alpha + i\beta} = e^{(\alpha + i\beta)\,\text{Log}\,(x + iy)} = e^{(\alpha + i\beta)[2n\pi i + \log(x + iy)]}$

$$= e^{(\alpha + i\beta)[2n\pi i + \log r(\cos\theta + i\sin\theta)]}$$

$$= e^{(\alpha + i\beta)[2n\pi i + \log r e^{i\theta}]}$$

$$= e^{(\alpha + i\beta)[2n\pi i + \log r + i\theta]}$$

$$= e^{(\alpha+i\beta)[\log r + i(2n\pi+\theta)]}$$

$$= e^{[\alpha \log r - \beta(2n\pi+\theta)] + i[\beta \log r + \alpha(2n\pi+\theta)]}$$

$$= e^{P+iQ} = e^{P} . e^{iQ}$$

$$= e^{P}[\cos Q + i \sin Q] = A + iB$$

where $A = e^{P} \cos Q$, $B = e^{P} \sin Q$ such that

$$P = \alpha \log r - \beta(2n\pi+\theta) = \alpha \log \sqrt{x^2+y^2} - \beta\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

$$= \frac{1}{2}\alpha \log(x^2+y^2) - \beta\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

$$Q = \beta \log r + \alpha(2n\pi+\theta) = \frac{1}{2}\beta \log(x^2+y^2) + \alpha\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

ILLUSTRATIVE EXAMPLES

Example 1. If $i^{\alpha+i\beta} = \alpha + i\beta$, prove that $\alpha^2+\beta^2 = e^{-(4n+1)\beta\pi}$.

(P.U. 2002; G.N.D.U. 2006; H.P.U. 2010)

Sol. Here $\alpha + i\beta = i^{\alpha+i\beta} = e^{(\alpha+i\beta)\,\text{Log}\, i} = e^{(\alpha+i\beta)[2n\pi i + \log i]}$

$$= e^{(\alpha+i\beta)\left[2n\pi i + \log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]} = e^{(\alpha+i\beta)\left[2n\pi i + \log e^{i\frac{\pi}{2}}\right]}$$

$$= e^{(\alpha+i\beta)\left[2n\pi i + i\frac{\pi}{2}\right]} = e^{-\beta(4n+1)\frac{\pi}{2} + i(4n+1)\frac{\pi}{2}\alpha}$$

$$= e^{-\beta(4n+1)\frac{\pi}{2}} . e^{i(4n+1)\frac{\pi}{2}\alpha}$$

$$\therefore \quad \alpha + i\beta = e^{-\beta(4n+1)\frac{\pi}{2}}\left[\cos(4n+1)\frac{\pi}{2}\alpha + i\sin(4n+1)\frac{\pi}{2}\alpha\right]$$

Equating real and imaginary parts,

$$\alpha = e^{-\beta(4n+1)\frac{\pi}{2}} . \cos(4n+1)\frac{\pi}{2}\alpha$$

and $$\beta = e^{-\beta(4n+1)\frac{\pi}{2}} . \sin(4n+1)\frac{\pi}{2}\alpha$$

Squaring and adding (1) and (2),

$$\alpha^2+\beta^2=e^{-\beta(4n+1)\pi}\left[\cos^2(4n+1)\frac{\pi}{2}\alpha+\sin^2(4n+1)\frac{\pi}{2}\alpha\right]$$

$$\therefore\quad \alpha^2+\beta^2=e^{-\beta(4n+1)\pi}.$$

Example 2. Prove that i^i is wholly real and find its principal value. Also show that the values of i^i form a G.P.

(Pbi. U. 2009; H.P.U. 2011)

Sol. We have $i^i=e^{i\,\mathrm{Log}\,i}=e^{i(2n\pi i+\log i)}$

$$=e^{i\left[2n\pi i+\log\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\right]}=e^{i\left(2n\pi i+\log e^{i\frac{\pi}{2}}\right)}=e^{i\left(2n\pi i+i\frac{\pi}{2}\right)}$$

$$=e^{-\left(2n\pi+\frac{\pi}{2}\right)}\text{, which is wholly real.}$$

Putting $n=0$, the principal value of i^i is $e^{-\frac{\pi}{2}}$.

Putting $n=0,1,2,\ldots$, the value of i^i are $e^{-\frac{\pi}{2}},\ e^{-\frac{5\pi}{2}},\ e^{-\frac{9\pi}{2}},\ \ldots$

which form a G.P. with common ratio $e^{-2\pi}$.

Example 3. If the principal values are considered, prove that

$$\frac{(1+i)^{1-i}}{(1-i)^{1+i}}=\sin(\log 2)+i\cos(\log 2).$$

(G.N.D.U. 2005)

Sol. Here we are to take only principal values

$$\therefore\quad \frac{(1+i)^{1-i}}{(1-i)^{1+i}}=\frac{e^{(1-i)\log(1+i)}}{e^{(1+i)\log(1-i)}}=\frac{e^{(1-i)\left[\frac{1}{2}\log(1+1)+i\tan^{-1}\frac{1}{1}\right]}}{e^{(1+i)\left[\frac{1}{2}\log(1+1)-i\tan^{-1}\frac{1}{1}\right]}}$$

$$\left[\because\ \log(a+ib)=\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}\frac{b}{a}\right]$$

$$=\frac{e^{(1-i)\left[\log\sqrt{2}+i\frac{\pi}{4}\right]}}{e^{(1+i)\left[\log\sqrt{2}-i\frac{\pi}{4}\right]}}=\frac{e^{\left(\log\sqrt{2}+\frac{\pi}{4}\right)+i\left(\frac{\pi}{4}-\log\sqrt{2}\right)}}{e^{\left(\log\sqrt{2}+\frac{\pi}{4}\right)-i\left(\frac{\pi}{4}-\log\sqrt{2}\right)}}$$

$$= e^{2i\left(\frac{\pi}{4} - \log\sqrt{2}\right)} = e^{i\left(\frac{\pi}{2} - 2\log\sqrt{2}\right)} = e^{i\left(\frac{\pi}{2} - \log 2\right)}$$

$$= \cos\left(\frac{\pi}{2} - \log 2\right) + i\sin\left(\frac{\pi}{2} - \log 2\right) = \sin(\log 2) + i\cos(\log 2)$$

= R.H.S.

Example 4. If $(a + ib)^p = m^{x+iy}$, then prove that $\frac{y}{x} = \frac{2\tan^{-1}\frac{b}{a}}{\log(a^2+b^2)}$ when only principal value is considered.

(G.N.D.U. 2010)

Sol. $(a+ib)^p = m^{x+iy}$

$\Rightarrow \quad \log(a+ib)^p = \log m^{x+iy}$

$\Rightarrow \quad p\log(a+ib) = (x+iy)\log m$

$\Rightarrow \quad p\left[\frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a}\right] = (x+iy)\log m$

$\Rightarrow \quad \frac{1}{2}p\log(a^2+b^2) + ip\tan^{-1}\frac{b}{a} = x\log m + iy\log m$

Equating real and imaginary parts,

$$x\log m = \frac{1}{2}p\log(a^2+b^2) \qquad ...(1)$$

$$y\log m = p\tan^{-1}\frac{b}{a} \qquad ...(2)$$

Dividing (2) by (1), we get, $\frac{y}{x} = \frac{2\tan^{-1}\frac{b}{a}}{\log(a^2+b^2)}$.

EXERCISE 14 (*d*)

1. Find the real part of $i^{\log(1+i)}$. (Consider only principal values)
2. Prove that

   (*i*) $(-i)^{-i} = e^{(4n-1)\frac{\pi}{2}}$ (Pbi. U. 2008)

   (*ii*) $i^x = \cos(4n+1)\frac{\pi}{2}x + i\sin(4n+1)\frac{\pi}{2}x$

   (*iii*) $x^i = e^{-2n\pi}[\cos(\log x) + i\sin(\log x)]$

$) - q \tan^{-1} \frac{y}{x} \log_q e$

and $\log_a (x^2 + y^2) = 2 \log \frac{p\alpha + q\beta}{p^2 + q^2}$

10. If $\tan \log (x + iy) = a + ib$ and $a^2 + b^2 \neq 1$, then prove that (Consider only principal values)

$$\tan \log (x^2 + y^2) = \frac{2a}{1 - a^2 - b^2}.$$

11. If $i^{i^{\cdots \infty}} = A + iB$ and only principal values are considered, prove that

(*i*) $\tan \frac{\pi A}{2} = \frac{B}{A}$

(*ii*) $A^2 + B^2 = e^{-\pi B}$ (P.U. 2002, 2008)

(P.U. 2008; Pbi. U. 2009)

ANSWERS

1. $e^{-\frac{\pi^2}{8}} \cos\left(\frac{\pi}{4} \log 2\right)$

4. $a = -1, b = 0 ; 0$

Art-17. Gregory's Series

If $-1 \leq x \leq 1$, then $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \infty$

(G.N.D.U. 2008)

Or

If $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, then $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots \infty$

(G.N.D.U. 2008, 2009, 2010)

Proof : We know that

$$\log(1 + ix) = \frac{1}{2}\log(1 + x^2) + i\tan^{-1} x \quad \ldots(1)$$

Also $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

Replacing x by ix, we get,

$$\log(1 + ix) = ix - \frac{i^2 x^2}{2} + \frac{i^3 x^3}{3} - \frac{i^4 x^4}{4} + \ldots$$

or $\log(1+ix) = ix + \frac{x^2}{2} - i\frac{x^3}{3} - \frac{x^4}{4} + ...$

$\therefore \quad \log(1+ix) = \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} ...\right) + i\left(x - \frac{x^3}{3} + \frac{x^5}{5} ...\right)$

From (1) and (2), we get,

$$\frac{1}{2}\log(1+x^2) + i\tan^{-1}x = \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} ...\right) + i\left(x - \frac{x^3}{3} + \frac{x^5}{5} ...\right)$$

Equating imaginary parts, we get,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - ..... \infty$$

Put $\tan^{-1}x = \theta$ or $x = \tan\theta$

$$\therefore \quad \theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - .... \infty$$

Cor. 1. We have

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} .....$$

Put $x = 1$

$$\therefore \quad \tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} ....$$

Cor. 2. $\therefore \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} ......$

Art-18. Use Gregory's series to find value of π.

(H.P.U. 2005; Pbi. U. 2009; G.N.D.U 2010)

Proof : Consider $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$

$$= \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

$$\therefore \quad \frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

$$\Rightarrow \quad \frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{5}\cdot\frac{1}{2^5}....\right) + \left(\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5}....\right)$$

$$\Rightarrow \quad \frac{\pi}{4} = 0{\cdot}7854 \text{ nearly}$$

$$\Rightarrow \quad \pi = 3{\cdot}1416 \text{ nearly}$$

Note 1. This method used to find value of π is due to **Euler's Series.**

2. There are two other series, which help us in finding the value of π.

Machine's Series : We have

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 4\left[\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - ...\right] - \left[\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{(239)^3} + \frac{1}{5(239)^5} - ...\right]$$

$$= 0{\cdot}7853983 \text{ nearly}$$

$$\therefore \quad \pi = 3{\cdot}14159$$

Rutherford's Series : We have

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{10} + \tan^{-1}\frac{1}{99}$$

$$\therefore \quad \pi = 3.14161 \text{ nearly}$$

Note 3. Whenever we are to find the sum of a given series, we arrange it in the form

$$x - \frac{x^3}{3} + \frac{x^5}{5} - ....$$

and use the result

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - ....$$

ILLUSTRATIVE EXAMPLES

Example 1. Prove that $1 - \frac{1}{3^2} + \frac{1}{5\cdot3^2} - \frac{1}{7\cdot3^3} + ... = \frac{\pi}{2\sqrt{3}}$.

(Pbi. U. 2010)

Sol. L.H.S. $= 1 - \frac{1}{3^2} + \frac{1}{5\cdot3^2} - \frac{1}{7\cdot3^3} + ... = 1 - \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + ...$

$$= \sqrt{3}\left[\frac{1}{\sqrt{3}} - \frac{1}{3}\cdot\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{\sqrt{3}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{3}}\right)^7 + \ldots\right]$$

$$= \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad [\because \text{ of Gregory's Series}]$$

$$= \sqrt{3}\cdot\frac{\pi}{6} = \frac{\pi}{2\sqrt{3}} = \text{R.H.S.}$$

Example 2. Prove that $\frac{\pi}{8} = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \ldots$

(Pbi. U. 2010)

Sol. R.H.S. $= \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \ldots = \frac{1}{2}\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \ldots\right]$

$$= \frac{1}{2}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right] = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8} = \text{L.H.S.}$$

Example 3. Prove that $\left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) \ldots = \frac{\pi}{4}$.

(H.P.U. 2005; P.U. 2005; Pbi. U. 2007)

Sol. L.H.S. $= \left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) + \ldots$

$$= \left(\frac{2}{3} - \frac{1}{3}\cdot\frac{2}{3^3} + \frac{1}{5}\cdot\frac{2}{3^5} - \ldots\right) + \left(\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{7^3} + \frac{1}{5}\cdot\frac{1}{7^5} - \ldots\right)$$

$$= 2\left[\frac{1}{3} - \frac{1}{3}\cdot\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \ldots\right] + \left[\frac{1}{7} - \frac{1}{3}\cdot\left(\frac{1}{7}\right)^3 + \frac{1}{5}\left(\frac{1}{7}\right)^5 - \ldots\right]$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}\right) + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4}\times\frac{1}{7}} = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$$

...ample 4. Prove that

$$\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=n\pi+\frac{\pi}{4}+\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\ldots$$

...l. L.H.S. $=\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=\tan^{-1}\left[\frac{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\sin\theta}}{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}\right]=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$

$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]=n\pi+\frac{\pi}{4}+\theta$ [On generalising]

$=n\pi+\frac{\pi}{4}+\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta\ \ldots=$ R.H.S.

EXERCISE 14 (*e*)

1. Prove that $\sqrt{2}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]=\frac{\pi}{2\sqrt{2}}$.

(Pbi. U. 2010)

2. Using Gregory's Series, prove that $1-\frac{1}{3\cdot4^2}+\frac{1}{5\cdot4^4}-\ldots=4\tan^{-1}\frac{1}{4}$.

(G.N.D.U. 2007; Pbi. U. 2013)

3. Using Gregory's Series, show that

$$\left(1-\frac{1}{3^{\frac{1}{2}}}\right)-\frac{1}{3}\left(1-\frac{1}{3^{\frac{3}{2}}}\right)+\frac{1}{5}\left(1-\frac{1}{3^{\frac{5}{2}}}\right)\ldots=\frac{\pi}{12}.$$

4. Prove that $\frac{\pi}{4}=\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right)-\ldots$

5. Show that

$$\frac{\pi}{4}=\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{8}\right)-\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{5^3}+\frac{1}{8^3}\right)+\frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{5^5}+\frac{1}{8^5}\right)-\ldots$$

(G.N.D.U. 2010, 2012)

6. If $-(\sqrt{2}-1)<x<\sqrt{2}-1$, show that

$$2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right)=\frac{2x}{1-x^2}-\frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3+\frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5-\ldots$$

Art-19. Hyperbolic Functions

If x be any number, real or complex, then we define

$$\sinh x \text{ (to be read as hyperbolic sine of } x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh x \text{ (to be read as hyperbolic cosine of } x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \qquad \text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Cor. $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0,\qquad \cosh 0 = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1+1}{2} = 1$

Note : $\cosh x + \sinh x = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2} = \dfrac{e^x + e^{-x} + e^x - e^{-x}}{2}$

$$= \frac{2e^x}{2} = e^x$$

and $\cosh x - \sinh x = \dfrac{e^x + e^{-x}}{2} - \dfrac{e^x - e^{-x}}{2} = \dfrac{e^x + e^{-x} - e^x + e^{-x}}{2} = \dfrac{2e^{-x}}{2} = e^{-x}$

Art-20. Prove that

(*i*) $\sinh x = x + \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} + \ldots$ (*ii*) $\cosh x = 1 + \dfrac{x^2}{\lfloor 2} + \dfrac{x^4}{\lfloor 4} + \ldots$

Proof. (*i*) L.H.S. $= \sinh x = \dfrac{1}{2}(e^x - e^{-x})$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \ldots\right) - \left(1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} - \frac{x^5}{\lfloor 5} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{\lfloor 3} + \frac{2x^5}{\lfloor 5} + \ldots\right] = x + \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} + \ldots$$

= R.H.S.

(*ii*) L.H.S. $= \cosh x = \dfrac{1}{2}(e^{x} + e^{-x})$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{\lfloor\!\underline{2}} + \frac{x^3}{\lfloor\!\underline{3}} + \frac{x^4}{\lfloor\!\underline{4}} + \frac{x^5}{\lfloor\!\underline{5}} + \ldots\right) + \left(1 - x + \frac{x^2}{\lfloor\!\underline{2}} - \frac{x^3}{\lfloor\!\underline{3}} + \frac{x^4}{\lfloor\!\underline{4}} - \frac{x^5}{\lfloor\!\underline{5}} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[2 + 2.\frac{x^2}{\lfloor\!\underline{2}} + 2.\frac{x^4}{\lfloor\!\underline{4}} + \ldots\right] = 1 + \frac{x^2}{\lfloor\!\underline{2}} + \frac{x^4}{\lfloor\!\underline{4}} + \ldots$$

= R.H.S.

Art-21. Find the relation between hyperbolic and circular functions.

Proof : (*i*) We have $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{i\theta})$

Putting $\theta = ix$, we get,

$$\sin(ix) = \frac{1}{2i}(e^{i^2x} - e^{-i^2x}) = \frac{1}{2i}(e^{-x} - e^{x})$$

$$= \frac{i}{2i^2}\, i(e^{-x} - e^{x}) = \frac{1}{2}\, i(e^{x} - e^{-x})$$

$$= i.\frac{1}{2}(e^{x} - e^{-x})$$

$\therefore$ $\sin(ix) = i\sinh x$ (Pbi. U. 2010, 2011)

(*ii*) Again $\cos\theta = \dfrac{1}{2}(e^{i\theta} + e^{-i\theta})$

Putting $\theta = ix$, we get,

$$\cos ix = \frac{1}{2}\left(e^{i^2x} + e^{-i^2x}\right) = \frac{1}{2}(e^{-x} + e^{x}) = \frac{1}{2}(e^{x} + e^{-x})$$

$\therefore$ $\cos(ix) = \cosh x$ (Pbi. U. 2011)

(*iii*) $\tan(ix) = \dfrac{\sin(ix)}{\cos(ix)} = \dfrac{i\sinh x}{\cosh x} = i\tanh x$

(*iv*) $\coth(ix) = \dfrac{\cos(ix)}{\sin(ix)} = \dfrac{\cosh x}{i\sinh x} = \dfrac{i}{i^2}\coth x = -i\coth x$

(*v*) $\sec(ix) = \dfrac{1}{\cos(ix)} = \dfrac{1}{\cosh x} = \operatorname{sech} x$

(vi) $\text{cosec}\,(i\,x) = \dfrac{1}{\sin\,(i\,x)} = \dfrac{1}{i\sinh\,x} = -i\,\text{cosech}\,x.$

Note : (1) $\sinh\,(i\,x) = \dfrac{e^{ix} - e^{-ix}}{2} = i\,.\,\dfrac{e^{ix} - e^{-x}}{2i} = i\sin x$

(2) $\cosh\,(i\,x) = \dfrac{e^{ix} + e^{-ix}}{2} = \cos x$

(3) $\tanh\,(i\,x) = \dfrac{\sinh\,(i\,x)}{\cosh\,(i\,x)} = \dfrac{i\sin x}{\cos x} = i\tan x$

Art-22. Prove that

(a) Fundamental Formulae

(i) $\cosh^2 x - \sinh^2 x = 1$ (ii) $\text{sech}^2 x - \tanh^2 x = 1$

(iii) $\coth^2 x - \text{cosech}^2 x = 1$

(b) Addition and Subtraction Formulae

(i) $\sinh\,(x + y) = \sinh x \cosh y + \cosh x \sinh y$

(ii) $\sinh\,(x - y) = \sinh x \cosh y - \cosh x \sinh y$

(iii) $\cosh\,(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(iv) $\cosh\,(x - y) = \cosh x \cosh y + \sinh x \sinh y$

(v) $\tanh\,(x - y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

(vi) $\tanh\,(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

(c) Formulae for 2x and 3x

(i) $\sinh 2x = 2\sinh x \cosh x = \dfrac{2\tanh x}{1 - \tan^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

$= \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$

(iii) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$ (Pbi. U. 2010)

(iv) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(v) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(vi) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

(*d*) A, B Formulae

(*i*) $2 \sinh A \cosh B = \sinh (A + B) + \sinh (A - B)$

(*ii*) $2 \cosh A \sinh B = \sinh (A + B) - \sinh (A - B)$

(*iii*) $2 \cosh A \cosh B = \cosh (A + B) + \cosh (A - B)$

(*iv*) $2 \sinh A \sinh B = \cosh (A + B) - \cosh (A - B)$

(*e*) C, D Formulae

(*i*) $\sinh C + \sinh D = 2 \sinh \frac{C+D}{2} \cosh \frac{C-D}{2}$

(*ii*) $\sinh C - \sinh D = 2 \cosh \frac{C+D}{2} \sinh \frac{C-D}{2}$

(*iii*) $\cosh C + \cosh D = 2 \cosh \frac{C+D}{2} \cosh \frac{C-D}{2}$

(*iv*) $\cosh C - \cosh D = 2 \sinh \frac{C+D}{2} \sinh \frac{C-D}{2}$

Proof : (*a*) (*i*) L.H.S. $= \cosh^2 x - \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = \frac{4}{4} = 1 = \text{R.H.S.}$$

Another Method :

We have $\sin^2 \theta + \cos^2 \theta = 1$. Put $\theta = i x$

$\therefore \quad \sin^2 (i x) + \cos^2 (i x) = 1 \quad$ or $\quad (i \sinh x)^2 + (\cosh x)^2 = 1$

$\therefore \quad -\sinh^2 x + \cosh^2 x = 1 \quad$ or $\quad \cosh^2 x - \sinh^2 x = 1$

(*b*) (*i*) L.H.S. $= \sinh (x + y)$

$$= \frac{1}{i} \sin i (x + y) = \frac{1}{i} \sin (i x + i y)$$

$$= \frac{i}{i^2} [\sin (i x) \cos (i y) + \cos (i x) \sin (i y)]$$

$$= -i [i \sinh x \cosh y + i \cosh x \sinh y] = \sinh x \cosh y + \cosh x \sinh y$$

$= \text{R.H.S.}$

(*c*) (*i*) We have $\sin 2\theta = 2 \sin \theta \cos \theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$. Put $\theta = i x$

$\therefore \quad \sin(2ix) = 2\sin(ix)\cos(ix) = \dfrac{2\tan(ix)}{1+\tan^2(ix)}$

or $\quad i\sinh 2x = 2i\sinh x\cosh x = \dfrac{2i\tanh x}{1-\tanh^2 x}$

$\therefore \quad \sinh 2x = 2\sinh x\cosh x = \dfrac{2\tanh x}{1-\tanh^2 x}$

(*d*) (*i*) We have

$2\sin x\cos y = \sin(x+y)+\sin(x-y).$

Put $\quad x = iA, y = iB$

$\therefore \quad 2\sin(iA)\cos(iB) = \sin(iA+iB)+\sin(iA-iB)$

$\because \quad 2i\sinh A\cosh B = i\sinh(A+B)+i\sinh(A-B)$

$\therefore \quad 2\sinh A\cosh B = \sinh(A+B)+\sinh(A-B)$

(*e*) (*i*) We have

$\sin x+\sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$. Put $x = iC, y = iD$

$\therefore \quad \sin(iC)+\sin(iD) = 2\sin\dfrac{iC+iD}{2}\cos\dfrac{iC-iD}{2}$

$\because \quad i\sinh C+i\sinh D = 2i\sinh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$

$\therefore \quad \sinh C+\sinh D = 2\sinh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$

Note 1. On the same lines, students can prove the remaining results.

Note 2. All the above results are obtained from circular identities by changing $\sin z$ to $i\sinh z$, $\cos z$ to $\cosh z$, $\tan z$ to $i\tanh z$.

Art-23. Prove that

(*i*) $\sinh\theta$ is periodic with period $2\pi i$.

(*ii*) $\cosh\theta$ is periodic with period $2\pi i$

(*iii*) $\tanh\theta$ is periodic with period πi

(*iv*) $\coth\theta$ is periodic with period πi

(*v*) $\text{sech}\,\theta$ is periodic with period $2\pi i$

(*vi*) $\text{cosech}\,\theta$ is periodic with period $2\pi i$

Proof. (*i*) We have

$$\sinh\theta = \frac{1}{2}(e^{\theta}-e^{-\theta})$$

Now $\sinh(\theta + 2n\pi i) = \frac{1}{2}\left[e^{\theta+2n\pi i} - e^{-(\theta+2n\pi i)}\right]$

$$= \frac{1}{2}\left[e^{\theta}.e^{2n\pi i} - e^{-\theta}e^{-2n\pi i}\right]$$

$$= \frac{1}{2}[e^{\theta}.1 - e^{-\theta}.1] \qquad \left[\because e^{2n\pi i} = e^{-2n\pi i} = 1\right]$$

$$= \frac{1}{2}(e^{\theta} - e^{-\theta})$$

$\therefore$ $\sinh(\theta + 2n\pi i) = \sinh\theta$ where $n \in N$

$\therefore$ $\sinh\theta$ remains unchanged when θ is increased by any multiple of $2\pi i$

$\therefore$ $\sinh\theta$ is periodic with periodic $2\pi i$

Note. Students can prove the remaining parts on the same lines.

ILLUSTRATIVE EXAMPLES

Example 1. Separate into real and imaginary parts :

(*i*) $\tan(x + iy)$ (*ii*) $\text{cosec}(x + iy)$ (*iii*) $\sinh(x + iy)$

(Pbi. U. 2009)

(*iv*) $\log\sin(x + iy)$ (Pbi. U. 2011)

Sol. (*i*) $\tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)} = \dfrac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

(*ii*) $\text{cosec}(x+iy) = \dfrac{1}{\sin(x+iy)} = \dfrac{2}{2\sin(x+iy)} \times \dfrac{\sin(x-iy)}{\sin(x-iy)}$

$$= \frac{2[\sin x\cos(iy) - \cos x\sin(iy)]}{\cos(2iy) - \cos 2x}$$

$$= \frac{2\sin x\cosh y - 2i\cos x\sinh y}{\cosh 2y - \cos 2x}$$

$$= \frac{2\sin x\cosh y}{\cosh 2y - \cos 2x} - i\frac{2\cos x\sinh y}{\cosh 2y - \cos 2x}$$

(iii) $\sinh(x + iy) = \frac{1}{i} \sin i(x + iy) = \frac{i}{i^2} \sin(ix - y)$

$= -i[\sin(ix)\cos y - \cos(ix)\sin y]$

$= -i[i \sinh x \cos y - \cosh x \sin y]$

$= \sinh x \cos y + i \cosh x \sin y$

(iv) $\log \sin(x + iy) = \log[\sin x \cos(iy) + \cos x \sin(iy)]$

$= \log[\sin x \cosh y + i \cos x \sinh y] = \log(\alpha + i\beta)$

where $\alpha = \sin x \cosh y, \beta = \cos x \sinh y$

$\therefore \quad \log \sin(x + iy) = \frac{1}{2} \log(\alpha^2 + \beta^2) + i \tan^{-1} \frac{\beta}{\alpha}$

$= \frac{1}{2} \log(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y) + i \tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right)$

$= \frac{1}{2} \log\left[\frac{1-\cos 2x}{2} \cdot \frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2} \cdot \frac{\cosh 2y - 1}{2}\right] + i \tan^{-1}(\cot x \tanh y)$

$= \frac{1}{2} \log\left[\frac{\cosh 2y - \cos 2x}{2}\right] + i \tan^{-1}(\cot x \tanh y)$

Example 2. Prove that $\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = 2i \tan^{-1}(\cot x \tanh y).$

(G.N.D.U. 2007)

Sol. We have

$\sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$

$= \sin x \cosh y + i \cos x \sinh y = \alpha + i\beta$

where $\quad \alpha = \sin x \cosh y, \beta = \cos x \sinh y$

$\therefore \quad \log \sin(x + iy) = \log(\alpha + i\beta) = \frac{1}{2} \log(\alpha^2 + \beta^2) + i \tan^{-1} \frac{\beta}{\alpha}$

$\therefore \quad \log \sin(x + iy) = \frac{1}{2} \log(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)$

$+ i \tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right) \quad \ldots(1)$

Changing i to $-i$, we get,

$\log \sin(x - iy) = \frac{1}{2} \log(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)]$

$- i \tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right) \quad \ldots(2)$

$$\text{L.H.S.} = \log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = \log\sin(x+iy) - \log\sin|x-iy|$$

$$= 2\,i\tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right) \qquad [\because \text{ of } (1), (2)]$$

$$= \text{R.H.S.}$$

Example 3. If $\sin(u+iv) = x+iy$, then

(*i*) $$\frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1$$

(G.N.D.U. 2009; H.P.U. 2009; P.U. 2010)

(*ii*) $$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1$$ (G.N.D.U. 2009; P.U. 2010)

Sol. $\because$ $x+iy = \sin(u+iv)$

$\therefore$ $x+iy = \sin u \cos(iv) + \cos u \sin(iv)$

$\therefore$ $x+iy = \sin u \cosh v + i\cos u \sinh v$

Equating real and imaginary parts,

$$x = \sin u \cosh v \quad \text{...(1)}$$

$$y = \cos u \sinh v \quad \text{...(2)}$$

From (1), $\sin u = \dfrac{x}{\cosh v}$

From (2), $\cos u = \dfrac{y}{\sinh v}$

Putting values of $\sin u$, $\cos u$ in $\sin^2 u + \cos^2 u = 1$, we get,

$$\frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1,$$ which proves result (*i*).

(*ii*) Again from (1), $\cosh v = \dfrac{x}{\sin u}$

From (2), $\sinh v = \dfrac{y}{\cos u}$

Putting values of $\cosh v$, $\sinh v$ in $\cosh^2 v - \sinh^2 v = 1$, we get

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1,$$ which is result (*ii*).

Example 4. If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$, where the letters denote real quantities, prove that,

(i) $\theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$ (P.U. 2005; Pbi. U. 2011)

(ii) $e^{2\phi} = \tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$ or $\phi = \dfrac{1}{2}\log\tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$

(P.U. 2005; Pbi. U. 2008; H.P.U. 2009; G.N.D.U. 2009)

Sol. (i) Here $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$ ...(1)

Changing i to $-i$ in (1), we get,

$\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$ ...(2)

Now $\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)] = \dfrac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}$

$$\therefore \quad \cot 2\theta = \frac{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}{\tan(\theta + i\phi) + \tan(\theta - i\phi)}$$

$$= \frac{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}{(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)} \quad [\because \text{ of } (1), (2)]$$

$$= \frac{1 - (\cos^2\alpha - i^2\sin^2\alpha)}{2\cos\alpha} = \frac{1 - (\cos^2\alpha + \sin^2\alpha)}{2\cos\alpha} = \frac{1-1}{2\cos\alpha}$$

$\therefore \quad \cot 2\theta = 0 \quad \Rightarrow \quad \cot 2\theta = \cot\dfrac{\pi}{2}$

$\therefore \quad 2\theta = n\pi + \dfrac{\pi}{2} \quad \Rightarrow \quad \theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$, where $n \in \mathbf{I}$

(ii) Again $\tan(2i\phi) = \tan[(\theta + i\phi) - (\theta - i\phi)]$

$$\therefore \quad i\tanh 2\phi = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta + i\phi)}$$

$$= \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)} \quad [\because \text{ of } (1), (2)]$$

$$= \frac{2i\sin\alpha}{1 + \cos^2\alpha + \sin^2\alpha} = \frac{2i\sin\alpha}{1+1}$$

$\therefore \quad i\tanh 2\phi = i\sin\alpha$

$\therefore \quad \tanh 2\phi = \sin\alpha$

or $\quad \dfrac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \dfrac{\sin\alpha}{1}$

Applying componendo and dividendo,

$$\frac{(e^{2\phi}-e^{-2\phi})+(e^{2\phi}+e^{-2\phi})}{(e^{2\phi}-e^{-2\phi})-(e^{2\phi}+e^{-2\phi})} = \frac{\sin\alpha+1}{\sin\alpha-1} \quad \text{or} \quad \frac{2.e^{2\phi}}{2e^{-2\phi}} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

$$\therefore \quad e^{4\phi} = \frac{1+\sin\alpha}{1-\sin\alpha} = \frac{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}-2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}$$

$$= \frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}$$

$$\therefore \quad e^{4\phi} = \left[\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right]^2 \quad \Rightarrow \quad e^{2\phi} = \frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}$$

$$\therefore \quad e^{2\phi} = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

Example 5. Find all the roots of the equation

$$\sin z = \cosh 4.$$

Sol. The given equation is

$$\sin z = \cosh 4$$

or $\quad \sin z = \cos i4 \qquad [\because \cos ix = \cosh x]$

or $\quad \sin z = \sin\left(\frac{\pi}{2}-4i\right)$

$$\therefore \quad z = n\pi + (-1)^n\left(\frac{\pi}{2}-4i\right) \text{ where } n \in I$$

$$[\because \sin\theta = \sin\theta \Rightarrow \theta = n\pi + (-1)^n\alpha]$$

Example 6. Find all the roots of the equation $\cosh z = \frac{1}{2}$

Sol. The given equation is

$$\cosh z = \frac{1}{2}$$

or $\quad \frac{e^z+e^{-z}}{2} = \frac{1}{2} \quad$ or $\quad e^z+e^{-z} = 1$

$\therefore \quad e^z + \frac{1}{e^z} = 1 \qquad \text{or} \qquad e^{2z} + 1 = e^z$

$\therefore \quad e^{2z} - e^z + 1 = 0$

Put $e^z = t$

$\therefore \quad t^2 - t + 1 = 0$

$$\therefore \quad t = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore \quad e^z = \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore \quad z = 2\,i\,n\pi + \log\left(\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}\right), \text{ where } n \in \mathrm{I}$$

$$= 2\,i\,n\pi + \log\left(\sqrt{\frac{1}{4} + \frac{3}{4}}\right) + i\tan^{-1}\sqrt{3} = 2\,i\,n\pi + \log 1 + i\,\frac{\pi}{3}$$

$$\therefore \quad z = i\left(2\,n\pi + \frac{\pi}{3}\right), \text{ where } n \in \mathrm{I}$$

Example 7. Find all values of z such that $\sinh z = e^{\frac{\pi i}{3}}$

Sol. The given equation is

$$\sinh z = e^{\frac{\pi i}{3}}$$

or $\quad \sinh(x + iy) = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, where $z = x + iy$

$$\therefore \quad \sinh x \cos y + i \cosh x \sin y = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}$$

Equating real and imaginary parts, we get,

$$\sinh x \cos y = \frac{1}{2} \quad \Rightarrow \quad \sinh x = \frac{1}{2\cos y} \qquad \text{...(1)}$$

and $\quad \cosh x \sin y = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \cosh x = \frac{\sqrt{3}}{2\sin y} \qquad \text{...(2)}$

From (1) from (2), we get,

$$\cosh^2 x - \sinh^2 x = \frac{3}{4\sin^2 y} - \frac{1}{4\cos^2 y}$$

or $$1 = \frac{3}{4} \cdot \frac{1}{\sin^2 y} - \frac{1}{4\cos^2 y}$$

$\therefore$ $$4\sin^2 y\cos^2 y = 3\cos^2 y - \sin^2 y$$

or $$4\sin^2 y(1-\sin^2 y) = 3(1-\sin^2 y) - \sin^2 y$$

$\therefore$ $$4\sin^2 y - 4\sin^4 y = 3 - 4\sin^2 y$$

$\therefore$ $$4\sin^4 y - 8\sin^2 y + 3 = 0$$

$\therefore$ $$\sin^2 y = \frac{8 \pm \sqrt{64-48}}{8} = \frac{8 \pm \sqrt{16}}{8} = \frac{8 \pm 4}{8} = \frac{12}{8}, \frac{4}{8} = \frac{3}{2}, \frac{1}{2}$$

Now $\sin^2 y = \frac{3}{2}$ is impossible as $\sin^2 y \le 1$

$$\sin^2 y = \frac{1}{2} \Rightarrow \sin y = \pm \frac{1}{\sqrt{2}}$$

Rejecting $\sin y = -\frac{1}{\sqrt{2}}$ as if $\sin y$ is –ve then from (2), $\cosh x$ is also –ve which is impossible

$\therefore$ $$\sin y = \frac{1}{\sqrt{2}} \quad \text{or} \quad \sin y = \sin\frac{\pi}{4}$$

$\therefore$ $$y = n\pi + (-1)^n \frac{\pi}{4} \text{ where } n \in \mathbf{I}$$

Two cases arise :

Case I. *n* is even.

$\therefore$ $$y = n\pi + \frac{\pi}{4}$$ and so $\cos y$ is positive as y is in first quadrant.

$\therefore$ $$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$\therefore$ from (1), $$\sinh x = \frac{1}{\sqrt{2}}$$

From (2), $$\cosh x = \frac{\sqrt{3}}{2 \cdot \frac{1}{\sqrt{2}}} = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}, \quad \therefore x = \cosh^{-1}\left(\sqrt{\frac{3}{2}}\right)$$

$\therefore$ $$x = \log\left(\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}-1}\right) \quad \left[\because \cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)\right]$$

$$\therefore \quad x = \log\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)$$

$$\therefore \quad z = x + iy = \log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) + i\left(n\pi + \frac{\pi}{4}\right)$$

Case II. n is odd.

$$\therefore \quad y = n\pi - \frac{\pi}{4} \text{ and } \cos y \text{ is negative as } \cos(n\pi - \theta) = -\cos\theta$$

$$\therefore \quad \cos y = \cos\left(n\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

From (1), $$\sinh x = \frac{1}{2\times\frac{-1}{\sqrt{2}}} = -\frac{1}{\sqrt{2}}$$

$$\therefore \quad x = \sinh^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \log\left(-\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2}+1}\right)$$

$$\left[\because \sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)\right]$$

$$\therefore \quad x = \log\left(-\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}\right) = \log\left(\frac{-1+\sqrt{3}}{\sqrt{2}}\right) = \log\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)$$

$$\therefore \quad z = x + iy = \log\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) + i\left(n\pi - \frac{\pi}{4}\right)$$

EXERCISE 14 (*f*)

1. Prove that (Pbi. U. 2011)

   (*i*) $\sinh(i\pi - \theta) = \sinh\theta$

   (*ii*) $\cosh(i\pi + \theta) = -\cosh\theta$ (*iii*) $\tanh\left(i\frac{\pi}{2} + \theta\right) = \coth\theta$