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APPLICATIONS OF DE MOIVRE'S THEOREM

Art-1. Roots of A Complex Number

Show that there are q and only q distinct values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$, q being a positive integer.

Proof : We know that $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$.

We want to find all the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$

$$(\cos\theta + i\sin\theta)^{\frac{1}{q}} = [\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]^{\frac{1}{q}}$$

$\therefore$ one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$ is $\cos\frac{2n\pi + \theta}{q} + i\sin\frac{2n\pi+\theta}{q}$

Putting $n = 0, 1, 2, \ldots, q-1$, we get

$$\text{1st value} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$$

$$\text{2nd value} = \cos\frac{2\pi + \theta}{q} + i\sin\frac{2\pi + \theta}{q}$$

$$\text{3rd value} = \cos\frac{4\pi + \theta}{q} + i\sin\frac{4\pi + \theta}{q}$$

....

$$q\text{th value} = \cos\frac{2(q-1)\pi + \theta}{q} + i\sin\frac{2(q-1)\pi + \theta}{q}$$

In these q values obtained, we find that angles involved in any two consecutive values differ from one another by $\frac{2\pi}{q} < 2\pi$ as q is a positive integer greater than 1.

Also we know that no two angles, which differ from one another by less than 2π, can have both their sines and cosines the same.

$\therefore$ all the q values obtained above are different.

Now putting $n = q$,

$$(q+1)\text{th value} = \cos\left(2\pi + \frac{\theta}{q}\right) + i\sin\left(2\pi + \frac{\theta}{q}\right) = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q} = \text{1st value}$$

Similarly, if we put $n = q+1, q+2, \ldots$, the values are repeated.

$\therefore$ $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$ has q and only q different values which are obtained by putting

$$n = 0, 1, 2, \ldots, q-1 \text{ in } \cos\frac{2n\pi+\theta}{q} + i\sin\frac{2n\pi+\theta}{q}.$$

Ex. 2. Show that $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ has q and only q distinct values, p and q being integers prime to each other.

(H.P.U. 2002, 2011; P.U. 2008; Pbi. U. 2009, 2011)

Proof: $(\cos\theta + i\sin\theta)^{\frac{p}{q}} = [(\cos\theta + i\sin\theta)^p]^{\frac{1}{q}} = (\cos p\theta + i\sin p\theta)^{\frac{1}{q}}$

[$\because$ p is an integer]

$$= [\cos(2n\pi + p\theta) + i\sin(2n\pi + p\theta)]^{\frac{1}{q}}$$

$\therefore$ one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ is $\cos\frac{2n\pi + p\theta}{q} + i\sin\frac{2n\pi+p\theta}{q}$

Putting $n = 0, 1, 2, \ldots q-1$ we get,

$$\text{1st value} = \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$$

$$\text{2nd value} = \cos\frac{2\pi + p\theta}{q} + i\sin\frac{2\pi+p\theta}{q}$$

$$\text{3rd value} = \cos\frac{4\pi + p\theta}{q} + i\sin\frac{4\pi+p\theta}{q}$$

....

$$q\text{th value} = \cos\frac{2(q-1)\pi + p\theta}{q} + i\sin\frac{2(q-1)\pi+p\theta}{q}$$

In these q values obtained, we find that angles involved in any two consecutive values differ from one another by $\frac{2\pi}{q} < 2\pi$ as q is a positive integer greater than 1.

Also we know that no two angles, which differ from one another by less than ... can have both their sines and cosines the same.

$\therefore$ all the q values obtained above are different.

Now putting $n = q$,

$$(q+1)\text{th value} = \cos\frac{2q\pi + p\theta}{q} + i\sin\frac{2q\pi + p\theta}{q}$$

$$= \cos\left(2\pi + \frac{p\theta}{q}\right) + i\sin\left(2\pi + \frac{p\theta}{q}\right)$$

$$= \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$$

Similarly, if we put $n = q+1,\ q+2, \ldots$, the values are repeated.

$\therefore$ $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ has q and only q different values which are obtained putting

$$n = 0, 1, 2, \ldots, q-1 \text{ in } \cos\frac{2n\pi + p\theta}{q} + i\sin\frac{2n\pi + p\theta}{q}$$

Art-3. Show that q values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ form a G.P. whose sum is zero, p, q being integers prime to each other.

(Pbi. U. 2009; H.P.U. 2...

Proof : $(\cos\theta + i\sin\theta)^{\frac{p}{q}} = [(\cos\theta + i\sin\theta)^p]^{\frac{1}{q}} = (\cos p\theta + i\sin p\theta)^{\frac{1}{q}}$

$$= [\cos(2n\pi + p\theta) + i\sin(2n\pi + p\theta)]^{\frac{1}{q}}$$

$$= \cos\frac{2n\pi + p\theta}{q} + i\sin\frac{2n\pi + p\theta}{q},$$

where $n = 0, 1, 2, \ldots, q$

$$= \cos\left(\frac{2n\pi}{q} + \frac{p\theta}{q}\right) + i\sin\left(\frac{2n\pi}{q} + \frac{p\theta}{q}\right)$$

$$= \left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)\left(\cos\frac{2n\pi}{q} + i\sin\frac{2n\pi}{q}\right)$$

$$= \left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)\left(\cos\frac{2\pi}{q} + i\sin\frac{2\pi}{q}\right)^n = ar^n$$

where $a = \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$, $r = \cos\frac{2\pi}{q} + i\sin\frac{2\pi}{q}$

Putting $n = 0, 1, 2, \ldots, q-1$, we get q roots of

$(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ as $a, ar, ar^2, \ldots, ar^{q-1}$ which from a G.P. whose first term is a and common ratio r.

$$\text{Sum of } q \text{ roots} = \frac{a(1-r^q)}{1-r} = \frac{a\left[1-\left(\cos\frac{2\pi}{q}+i\sin\frac{2\pi}{q}\right)^q\right]}{1-r}$$

$$= \frac{a[1-(\cos 2\pi + i\sin 2\pi)]}{1-r}$$

$$= \frac{a[1-(1+i.0)]}{1-r} = \frac{a[1-1]}{1-r} = \frac{0}{1-r} = 0 \qquad [\because r \neq 1]$$

Note. Important resultant to be remembered

1. $1 = \cos 0 + i\sin 0$
2. $-1 = \cos\pi + i\sin\pi$
3. $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
4. $-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

ILLUSTRATIVE EXAMPLES

Example 1. Prove that the n nth roots of unity form a series in G.P. Also show that their sum is zero and product is equal to $(-1)^{n-1}$.

(P.U. 2004, 2007, 2010 ; G.N.D.U. 2001; Pbi. U. 2008, 2011, 2012; H.P.U. 2012)

Sol. $(1)^{\frac{1}{n}} = (\cos 0 + i\sin 0)^{\frac{1}{n}} = [\cos(2r\pi + 0) + i\sin(2r\pi+0)]^{\frac{1}{n}}$

$$= \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n} \qquad \text{where } r = 0, 1, 2, 3, \ldots, n-1.$$

Putting $r = 0, 1, 2, 3, \ldots, n-1$, we get n roots as

$$(\cos 0 + i\sin 0), \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right), \left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right), \ldots, \left(\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}\right)$$

or $$1, \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right), \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^2, \ldots, \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n-1}$$

or $$1, t, t^2, t^3, \ldots, t^{n-1} \text{ where } t = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

These roots form a G.P.

Sum of roots $= 1 + t + t^2 + \dots + t^{n-1}$

$$= \frac{1(1-t^n)}{1-t} = \frac{1-\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n}{1-t}$$

$$= \frac{1-(\cos 2\pi + i\sin 2\pi)}{1-t} = \frac{1-(1+0)}{1-t} = \frac{0}{1-t} = 0$$

Product of roots $= 1.\, t.\, t^2 \dots t^{n-1}$

$$= t^{1+2+3+\dots+(n-1)} = t^{\frac{n-1}{2}[1+(n-1)]} = t^{\frac{n(n-1)}{2}}$$

$$= \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{\frac{n(n-1)}{2}}$$

$$= \cos\left\{\frac{2\pi}{n}.\frac{n(n-1)}{2}\right\} + i\sin\left\{\frac{2\pi}{n}.\frac{n(n-1)}{2}\right\}$$

$$= \cos(n-1)\pi + i\sin(n-1)\pi = [\cos\pi + i\sin\pi]^{n-1} = (-1)^{n-1}.$$

Note : Out of nth roots of unity, only one root namely 1 is real.

Example 2. Find the four fourth roots of $-1+i\sqrt{3}$.

(P.U. 2012)

Sol. Let $-1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts, we get

$r\cos\theta = -1$ ...(1)

$r\sin\theta = \sqrt{3}$ ...(2)

Squaring and adding (1) and (2), we get

$$r^2(\cos^2\theta + \sin^2\theta) = 1 + 3 \quad \text{or} \quad r^2 = 4 \quad \Rightarrow \quad r = 2$$

From (1) and (2), we get

$$\cos\theta = -\frac{1}{2},\ \sin\theta = \frac{\sqrt{3}}{2}$$

Both these equations are satisfied when $\theta = \frac{2\pi}{3}$

$$\therefore \quad -1+i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$\therefore \quad \left(-1+i\sqrt{3}\right)^{\frac{1}{4}} = 2^{\frac{1}{4}}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]^{\frac{1}{4}}$$

$$= 2^{\frac{1}{4}} \left[\cos\left(2n\pi + \frac{2\pi}{3}\right) + i \sin\left(2n\pi + \frac{2\pi}{3}\right)\right]^{1/4}$$

$$= 2^{\frac{1}{4}} \left[\cos\frac{(6n+2)\pi}{3} + i \sin\frac{(6n+2)\pi}{3}\right]^{1/4}$$

$$= 2^{\frac{1}{4}} \left(\cos\frac{(3n+1)\pi}{6} + i \sin\frac{(3n+1)\pi}{6}\right), \text{where } n = 0, 1, 2, 3.$$

Putting $n = 0, 1, 2, 3$, the required values are

$$2^{\frac{1}{4}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right),\ 2^{\frac{1}{4}}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right),\ 2^{\frac{1}{4}}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right),$$

$$2^{\frac{1}{4}}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

or $$2^{\frac{1}{4}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right), 2^{\frac{1}{4}}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right), 2^{\frac{1}{4}}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right),$$

$$2^{\frac{1}{4}}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

or $$2^{\frac{1}{4}}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right),\ 2^{\frac{1}{4}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right),\ 2^{\frac{1}{4}}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right),\ 2^{\frac{1}{4}}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

or $$2^{\frac{1}{4}}\left(\frac{\sqrt{3}+i}{2}\right),\ 2^{\frac{1}{4}}\left(\frac{-1+i\sqrt{3}}{2}\right),\ 2^{\frac{1}{4}}\left(\frac{-\sqrt{3}-i}{2}\right),\ 2^{\frac{1}{4}}\left(\frac{1-i\sqrt{3}}{2}\right)$$

Example 3. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that the continued product of all the values is 1.

(H.P.U. 2010)

Sol. Let $\frac{1}{2} + i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts,

$$r\cos\theta = \frac{1}{2} \quad ...(1)$$

$$r\sin\theta = \frac{\sqrt{3}}{2} \quad ...(2)$$

Squaring and adding (1) and (2), we get,

$$r^2 = \frac{1}{4}+\frac{3}{4} = 1, \quad \therefore \; r = 1$$

From (1) and (2), we get,

$$\cos\theta = \frac{1}{2}, \; \sin\theta = \frac{\sqrt{3}}{2}$$

Both these equations are satisfied when $\theta = \frac{\pi}{3}$

$$\therefore \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} = \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} = \left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3}\right]^{\frac{1}{4}}$$

$$= [\cos\pi + i\sin\pi]^{1/4}$$

$$= [\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)]^{1/4}$$

$$= \cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4} \quad \text{where } n = 0, 1, 2, 3$$

Putting $n = 0, 1, 2, 3$, the values are

$$\cos\frac{\pi}{4}+i\sin\frac{\pi}{4},\; \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4},\; \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4},\; \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$$

Continued product of all these values

$$= \left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

$$= \text{cis}\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right) = \text{cis}\, 4\pi = \cos 4\pi + i\sin 4\pi$$

$$= (\cos\pi + i\sin\pi)^4 = (-1)^4 = 1.$$

EXERCISE 13 (*a*)

1. Prove the following :

(*i*) Product of any two *n*th roots of unity is also an *n*th root of unity.

(G.N.D.U. 2001)

(*ii*) Reciprocal of any *n*th root of unity is also an *n*th root of unity.

2. Find the *n* *n*th roots of -1 and show that

(*i*) no two of these are equal and

(*ii*) any one root can be expressed as a power of any other.

(b) $2^{3/4}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),\ 2^{3/4}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right),$

$$2^{3/4}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right),\ 2^{3/4}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

(c) $\frac{\sqrt{3}+i}{2},\ \frac{-1+i\sqrt{3}}{2},\ \frac{-\sqrt{3}-i}{2},\ \frac{1-i\sqrt{3}}{2}$

$\left(2\sin\frac{\phi}{2}\right)^{1/3}\left(\cos\frac{\pi-\phi}{6}-i\sin\frac{\pi-\phi}{6}\right),\ \left(2\sin\frac{\phi}{2}\right)^{1/3}\left(\cos\frac{5\pi-\phi}{6}-i\sin\frac{5\pi-\phi}{6}\right),$

$$\left(2\sin\frac{\phi}{2}\right)^{1/3}\left(\cos\frac{9\pi-\phi}{6}-i\sin\frac{9\pi-\phi}{6}\right).$$

12. $2^{\frac{4}{3}}\cos\frac{\pi}{9},\ 2^{\frac{4}{3}}\cos\frac{7\pi}{9},\ 2^{\frac{3}{4}}\cos\frac{13\pi}{9}$

13. $\rho=\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}$;

$$\pm\left(\cos\frac{5\pi}{48}+i\sin\frac{5\pi}{48}\right),\ \pm\left(\cos\frac{29\pi}{48}+i\sin\frac{29\pi}{48}\right).$$

1-4. Primitive nth Root of Unity

A complex number z is called a primitive nth root of unity if and only if n is the least positive integer such that $z^n=1$.

In other words, z is a primitive nth root of unity iff

(*i*) $z^n = 1$ (*ii*) $z^m \neq 1$ for $m < n$, $m \in N$

Examples :

(*i*) Since 3 is the least positive integer such that $\omega^3 = 1$

$\therefore$ ω is a primitive cube root of unity.

(*ii*) Since 4 is the least positive integer such that $i^4 = 1$

$\therefore$ i is a primitive fourth root of unity.

(*iii*) Since $(-1)^1 = -1$, $(-1)^2 = 1$

$\therefore$ -1 is a primitive square root of unity.

Art-5. Prove that $\xi = \text{cis}\,\dfrac{2\pi}{n}$ is a primitive nth root of unity.

(G.N.D.U. 2001, 2002, 2003)

Proof : Here $\xi = \text{cis}\,\dfrac{2\pi}{n} = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$

$$\therefore \quad \xi^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = \cos 2\pi + i\sin 2\pi = 1 + i\,0 = 1$$

Now we will show that n is the least positive integer such that $\xi^n = 1$.

If possible suppose that m is a positive integer less than n, such that $\xi^m = 1$.

$$\therefore \quad \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^m = 1$$

$$\therefore \quad \cos\left(\frac{2\pi}{n}m\right) + i\sin\left(\frac{2\pi}{n}m\right) = 1$$

$$\therefore \quad \cos\left(\frac{m}{n}2\pi\right) + i\sin\left(\frac{m}{n}2\pi\right) = 1 \quad \Rightarrow \quad \frac{m}{n} \text{ is an integer}$$

$\Rightarrow$ $m \geq n$, which contradicts that $m < n$

$\therefore$ our supposition is wrong

$\therefore$ n is the least positive integer such that $\xi^n = 1$

$\therefore$ $\xi = \text{cis}\,\dfrac{2\pi}{n}$ is a primitive nth root of unity.

Art-6. If $\xi = \text{cis}\,\dfrac{2\pi}{n}$ and $k \in \mathbf{I}$ such that $(k, n) = d$, then ξ^k is a primitive $\left(\dfrac{n}{d}\right)$th root of unity.

(G.N.D.U. 2009, 2010)

Proof : Here $(k, n) = d$

$\therefore$ d/k and d/n

$\therefore$ there exist integers k_1, n_1 such that

$k = k_1 d$, $n = n_1 d$ and $(k_1, n_1) = 1$

We shall prove that $\xi^k = \left(\text{cis}\frac{2\pi}{n}\right)^k = \text{cis}\frac{2k\pi}{n}$ is a primitive n_1th root of unity.

(i) $(\xi^k)^{n_1} = (\xi^{k_1 d})^{n_1} = (\xi^{n_1 d})^{k_1} = (\xi^n)^{k_1}$

$$= \left(\left(\text{cis}\frac{2\pi}{n}\right)^n\right)^{k_1} = (\text{cis}\, 2\pi)^{k_1} = (1)^{k_1} = 1$$

(ii) If possible, suppose that $(\xi^k)^m = 1$, $m < n_1$ and $m \in \mathbf{N}$

$\therefore$ $\xi^{km} = 1$ $\Rightarrow$ $\left(\text{cis}\frac{2\pi}{n}\right)^{km} = 1$ $\Rightarrow$ $\text{cis}\frac{2km\pi}{n} = 1$

$\Rightarrow$ $\cos\left(\frac{km}{n}.2\pi\right) + i\sin\left(\frac{km}{n}.2\pi\right) = 1$

$\Rightarrow$ $\frac{km}{n}$ is an integer

$\Rightarrow$ n/km $\Rightarrow$ $n_1 d/k_1 dm$ $\Rightarrow$ $n_1/k_1 m$

But $(k_1, n_1) = 1$

$\therefore$ $n_1 | m$ $\Rightarrow$ $m \geq n_1$, which contradicts that $m < n_1$

$\therefore$ our supposition is wrong

$\therefore$ $(\xi^k)^m \neq 1$ for $m < n_1$, $m \in \mathbf{N}$

$\therefore$ ξ^k is a primitive n_1th *i.e.*, $\left(\frac{n}{d}\right)$th root of unity.

Cor. 1. If $\xi = \text{cis}\frac{2\pi}{n}$ and $k \in \mathbf{I}$ such that $(k, n) = 1$, then ξ^k is a primitive nth root of unity.

Proof : Since $(k, n) = 1$

$\therefore$ $d = 1$

$\therefore$ ξ^k is a primitive $\frac{n}{1}$th *i.e.*, nth root of unity.

Cor. 2. If $\xi = \text{cis}\frac{2\pi}{n}$, then ξ^2 is a primitive nth root of unity iff n is odd.

Proof : Take $k = 2$

$\therefore$ ξ^2 is a primitive nth root of unity iff $(2, n) = 1$ *i.e.*, n is odd.

ILLUSTRATIVE EXAMPLES

Example 1. Prove that the reciprocal of a primitive nth root of unity is also primitive nth root of unity.

Sol. Take $\xi = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

Let ξ^k be a primitive nth root of unity.

$\therefore \quad (k, n) = 1 \quad \Rightarrow (-k, n) = 1$

$\therefore \quad \xi^{-k}$ is a primitive nth root of unity

$\therefore \quad \frac{1}{\xi^k}$ *i.e.*, the reciprocal of ξ^k is also a primitive nth root of unity.

Example 2. Find the primitive 8th roots of unity.

(Pbi. U. 201)

Sol. Let $\xi = \text{cis}\,\frac{2\pi}{8} = \cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8}$

$$\therefore \quad \xi^8 = \left(\cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8}\right)^8 = \cos 2\pi + i\sin 2\pi = 1 + i\,0 = 1$$

$\therefore$ for a positive integer $k < 8$, ξ^k is a primitive 8th root of unity iff $(k, 8) = 1$.

$\therefore \quad k = 1, 3, 5, 7$

$\therefore \quad \xi, \xi^3, \xi^5, \xi^7$ are primitive 8th roots of unity.

Example 3. If ξ is a primitive nth root of unity, then

$$(x - \xi)(x - \xi^2)(x - \xi^3) \ldots (x - \xi^{n-1}) = x^{n-1} + x^{n-2} + \ldots + x + 1$$

Sol. Here $\xi = \text{cis}\,\frac{2\pi}{n}$ is a primitive nth root of unity

$\therefore \quad 1, \xi, \xi^2, \ldots, \xi^{n-1}$ are distinct n, nth roots of unity.

$\Rightarrow \quad 1, \xi, \xi^2, \ldots, \xi^{n-1}$ are n roots of the equation

$$x^n = 1 \quad \text{or} \quad x^n - 1 = 0$$

$$\therefore \quad x^n - 1 = (x - 1)(x - \xi)(x - \xi^2) \ldots\ldots\ldots\ldots (x - \xi^{n-1}) \quad \ldots(1)$$

Also $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \ldots + x + 1) \quad \ldots(2)$

From (1) and (2), we get,

$$(x - \xi)(x - \xi^2) \ldots (x - \xi^{n-1}) = x^{n-1} + x^{n-2} + \ldots + x + 1.$$

Example 4. If ξ is a primitive 12th root of unity, find $(x-\xi)(x-\xi^5)(x-\xi^7)(x-\xi^{11})$.

Sol. Let $\xi = \text{cis}\,\dfrac{2\pi}{12} = \cos\dfrac{2\pi}{12} + i\sin\dfrac{2\pi}{12}$

$\therefore$ ξ is a primitive 12th root of unity

and $\xi^{12} = \left(\cos\dfrac{2\pi}{12} + i\sin\dfrac{2\pi}{12}\right)^{12} = \cos 2\pi + i\sin 2\pi = 1 + i\,.\,0 = 1$

Now $\xi^5 = \left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)^5 = \cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}$

$$\xi^7 = \xi^{12}\,.\,\xi^{-5} = 1\,.\,\xi^{-5} = \xi^{-5} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{-5} = \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$$

$$\xi^{11} = \xi^{12}\,.\,\xi^{-1} = 1\,.\,\xi^{-1} = \xi^{-1} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{-1} = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$$

$$\therefore\ \xi^5 + \xi^7 = \left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right) = 2\cos\frac{5\pi}{6}$$

$$= 2\cos\left(\pi - \frac{\pi}{6}\right) = -2\cos\frac{\pi}{6} = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

and $$\xi + \xi^{11} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) + \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$\therefore\ (x-\xi)(x-\xi^5)(x-\xi^7)(x-\xi^{11}) = [(x-\xi)(x-\xi^{11})]\,[(x-\xi^5)(x-\xi^7)]$$

$$= [x^2 - (\xi + \xi^{11})x + \xi^{12}]\,[x^2 - (\xi^5 + \xi^7)x + \xi^{12}]$$

$$= (x^2 - \sqrt{3}\,x + 1)(x^2 + \sqrt{3}\,x + 1)$$

$$= (x^2+1)^2 - (\sqrt{3}\,x)^2 = x^4 + 2x^2 + 1 - 3x^2 = x^4 - x^2 + 1.$$

Example 5. Show that each primitive 6th root of unity satisfies $z^2 - z + 1 = 0$. (P.U. 2011)

Sol. Let $\xi = \text{cis}\,\dfrac{2\pi}{6} = \cos\dfrac{2\pi}{6} + i\sin\dfrac{2\pi}{6}$

$\therefore$ $\xi^6 = \left(\cos\dfrac{2\pi}{6} + i\sin\dfrac{2\pi}{6}\right)^6 = \cos 2\pi + i\sin 2\pi = 1 + i\,.\,0 = 1$

$\therefore$ for a positive integer $k < 6$, ξ^k is a primitive 6th root of unity iff $(k, 6) = 1$

$\therefore$ $k = 1, 5$

$\therefore$ ξ, ξ^5 are primitive 6th roots of unity.

Now $\xi^5 = \xi^6 . \xi^{-1} = 1 . \xi^{-1} = \xi^{-1} = \left(\cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}\right)^{-1} = \cos\frac{2\pi}{6} - i\sin\frac{2\pi}{6}$

$\therefore \quad \xi + \xi^5 = \left(\cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}\right) + \left(\cos\frac{2\pi}{6} - i\sin\frac{2\pi}{6}\right) = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$

The equation with ξ, ξ^5 as roots is $(z - \xi)(z - \xi^5) = 0$.

or $\quad z^2 - (\xi + \xi^5) z + \xi^6 = 0 \qquad$ or $\qquad z^2 - z + 1 = 0$

Hence the result.

EXERCISE 13 (*b*)

1. State which of the following statements are true of false. Justify your answer.
   (*i*) Every primitive *n*th root of unity is an *n*th root of unity.
   (*ii*) The sum of two primitive *n*th root of unity is also a primitive *n*th root of unity.
   (*iii*) The product of two primitive *n*th root of unity is also a primitive *n*th root of unity.
   (*iv*) The square of a primitive *n*th root of unity is also a primitive *n*th root of unity.
   (*v*) An integral power of a primitive *n*th root of unity is also primitive *n*th root of unity.
2. Give an example to show that if ξ is a primitive *n*th root of unity, then ξ^2 may not be a primitive *n*th root of unity.
3. If $\xi = \text{cis}\,\frac{2\pi}{7}$, express $\xi^{-1}, \xi^{-3}, \xi^{-13}, \xi^{-15}, \xi^{13}$ as ξ^k where $0 < k < 7$.
4. Find the primitive 6th roots of unity.
5. Find the primitive
   (*i*) 4th (*ii*) 6th (*iii*) 7th (*iv*) 9th (*v*) 10th (*vi*) 12th roots of unity.
6. Show that if p is a prime, then there are exactly $p-1$ primitive p^{th} roots of unity.
7. If ξ is a primitive 8th root of unity, find $(x - \xi)(x - \xi^3)(x - \xi^5)(x - \xi^7)$. (P.U. 2002)
8. Show that each primitive 8th root of unity satisfies $x^4 + 1 = 0$. (G.N.D.U. 2009; P.U. 2011, 2013)
9. Show that each primitive 12th root of unity satisfies $x^4 - x^2 + 1 = 0$.
10. Find an equation of least degree with real coefficients whose roots are the primitive 10th roots of unity.

1. (i) True (ii) False (iii) False (iv) False (v) False

2. $\xi^6, \xi^4, \xi, \xi^6, \xi^6$

3. ξ, ξ^5 where $\xi = \text{cis}\ \dfrac{2\pi}{6}$

4. ξ, ξ^5 where $\xi = \text{cis}\ \dfrac{2\pi}{6}$

5. (i) ξ, ξ^5 where $\xi = \text{cis}\ \dfrac{2\pi}{6}$

(ii) ξ, ξ^5 where $\xi = \text{cis}\ \dfrac{2\pi}{6}$

(iii) $\xi, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6$ where $\xi = \text{cis}\ \dfrac{2\pi}{7}$

(iv) $\xi, \xi^2, \xi^4, \xi^5, \xi^7, \xi^8$ where $\xi = \text{cis}\ \dfrac{2\pi}{9}$

(v) ξ, ξ^3, ξ^7, ξ^9 where $\xi = \text{cis}\ \dfrac{2\pi}{10}$

(vi) $\xi, \xi^5, \xi^7, \xi^{11}$ where $\xi = \text{cis}\ \dfrac{2\pi}{12}$

7. $x^4 + 1$

10. $x^4 - x^3 + x^2 - x + 1 = 0$

SOLUTIONS OF EQUATIONS

Art-7. There are certain types of equations in which use of De Moivre's theorem makes the solution very easy. We give some examples to illustrate this fact.

ILLUSTRATIVE EXAMPLES

Example 1. Solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$.

(H.P.U. 2005, 2011; Pbi. U. 2011)

Sol. The given equation is

$$x^4 - x^3 + x^2 - x + 1 = 0 \quad \text{...(1)}$$

Multiplying both sides by $x + 1$, we get,

$$x^5 + 1 = 0 \quad \text{...(2)}$$

or $\quad x^5 = -1$

$\therefore \quad x = (-1)^{1/5} = (\cos \pi + i \sin \pi)^{1/5}$

$= [\cos (2n\pi + \pi) + i \sin (2n\pi + \pi)]^{1/5}$

$= \cos \dfrac{(2n+1)\pi}{5} + i \sin \dfrac{(2n+1)\pi}{5}$ where $n = 0, 1, 2, 3, 4.$

Putting $n = 0, 1, 2, 3, 4$, the roots of (2) are

$$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5},\ \cos\pi + i\sin\pi,$$

$$\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5},\ \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$

or $\cos\frac{\pi}{5} + i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, -1,\ \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5},\ \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$

or $-1,\ \cos\frac{r\pi}{5} \pm i\sin\frac{r\pi}{5}$ where $r = 1, 3$

But the root -1 corresponds to the factor $x + 1$.

$\therefore$ roots of (1) are $\cos\frac{r\pi}{5} \pm i\sin\frac{r\pi}{5}$ where $r = 1, 3$.

Example 2. Solve the equation $x^{10} + x^8 + x^6 + x^4 + x^2 + 1 = 0$.

(G.N.D.U. 20..)

Sol. The given equation is $x^{10} + x^8 + x^6 + x^4 + x^2 + 1 = 0$ ...(1)

Multiplying both sides by $x^2 - 1$, we get

$$(x^2 - 1)(x^{10} + x^8 + x^6 + x^4 + x^2 + 1) = 0 \text{ or } x^{12} - 1 = 0$$ ...(2)

$\therefore$ $x^{12} = 1$ or $x = (1)^{\frac{1}{12}} = (\cos 0 + i\sin 0)^{\frac{1}{12}}$

$\therefore$ $x = [\cos(2n\pi + 0) + i\sin(2n\pi + 0)]^{\frac{1}{12}} = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{12}}$

$$= \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}, \text{ where } n = 0, 1, 2, 3, 4, 5, \ldots\ldots, 11.$$

Putting $n = 0, 1, 2, 3, \ldots\ldots, 11$, we get

$$x = \cos 0 + i\sin 0,\ \cos\frac{\pi}{6} + i\sin\frac{\pi}{6},\ \cos\frac{\pi}{3} + i\sin\frac{\pi}{3},\ \cos\frac{\pi}{2} + i\sin\frac{\pi}{2},$$

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6},\ \cos\pi + i\sin\pi,$$

$$\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3},\ \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2},$$

$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3},\ \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$$

or $x = 1,\ \cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6},\ \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3},\ i,\ \cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3},$

$\cos\frac{5\pi}{6} \pm i\sin\frac{5\pi}{6}, -1, -i$ are roots of (2)

$\therefore$ roots of (1) are

$$x = \cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}, \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}, \cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3},$$

$$\cos\frac{5\pi}{6} \pm i\sin\frac{5\pi}{6}, \pm i$$

[We have left $x = 1, -1$ which were introduced by multiplication of $x^2 - 1$]

Example 3. Solve $(1+z)^6 + z^6 = 0$ where $z \in \mathbf{C}$ is a complex number. (G.N.D.U. 2009)

Sol. The given equation is

$$(1+z)^6 + z^6 = 0 \quad \text{or} \quad (1+z)^6 = -z^6$$

$$\therefore \left(\frac{1+z}{z}\right)^6 = -1 = \cos\pi + i\sin\pi \quad i.e. \quad \frac{1+z}{z} = (\cos\pi + i\sin\pi)^{\frac{1}{6}}$$

$$\therefore \frac{1}{z} + 1 = [\cos(2n\pi + \pi) + i\sin(2n\pi \quad \pi)]^{\frac{1}{6}}$$

$$\therefore \frac{1}{z} = -1 + \left[\cos\frac{(2n+1)\pi}{6} + i\sin\frac{(2n+1)\pi}{6}\right]$$

$$\therefore \frac{1}{z} = -1 + \cos\theta + i\sin\theta, \text{ where } \theta = \frac{(2n+1)\pi}{6}$$

$$\therefore z = \frac{1}{-1 + \cos\theta + i\sin\theta} = \frac{1}{-2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{1}{2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \frac{1}{i}\sin\frac{\theta}{2}\right)} = -\frac{i}{2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}$$

$$= -\frac{i}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1} = -\frac{i}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$

$$= -\frac{1}{2}\left(i\cot\frac{\theta}{2} + 1\right)$$

$$\therefore z = -\frac{1}{2}\left(1 + i\cot\frac{\theta}{2}\right), \quad n = 0, 1, 2, 3, 4, 5.$$

Example 4. Prove that the roots of the equation $(x-1)^n = x^n$ (n being +ve integer) are $\frac{1}{2}\left[1+i\cot\frac{r\pi}{n}\right]$ where r has the values 0, 1, 2, ..., $n-1$.

(P.U. 2006, 2007, 2011; G.N.D.U. 2003; Pbi. U. 20..)

Sol. The given equation is

$$(x-1)^n = x^n \quad \text{or} \quad \left(\frac{x-1}{x}\right)^n = 1, \quad \text{or} \quad \frac{x-1}{x} = (1)^{1/n}$$

$$\therefore \quad \frac{x-1}{x} = (\cos 0 + i\sin 0)^{1/n} = [\cos(2r\pi + 0) + i\sin(2r\pi +)]^{1/n}$$

$$= \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}, \text{ where } r = 0, 1, 2, \ldots, n-1$$

$$\therefore \quad \frac{x-1}{x} = \frac{\cos\theta + i\sin\theta}{1}, \quad \text{where } \theta = \frac{2r\pi}{n}$$

$$\therefore \quad \frac{x}{x} - \frac{1}{x} = \cos\theta + i\sin\theta \quad \text{or} \quad 1 - \frac{1}{x} = \cos\theta + i\sin\theta$$

$$\therefore \quad \frac{1}{x} = 1 - \cos\theta - i\sin\theta$$

$$\therefore \quad x = \frac{1}{1-\cos\theta - i\sin\theta} = \frac{1}{2\sin^2\frac{\theta}{2} - i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{1}{-2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \frac{1}{i}\sin\frac{\theta}{2}\right)} = \frac{i}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1}$$

$$= \frac{i}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) = \frac{1}{2}\left[i\cot\frac{\theta}{2} + 1\right]$$

$$\therefore \quad x = \frac{1}{2}\left[1 + i\cot\frac{r\pi}{n}\right], \text{ where } r = 0, 1, 2, \ldots, n-1.$$

Another Form :

Solve $(z-1)^n = z^n$ where n is a positive integer and show that the real part of each root is $\frac{1}{2}$.

(G.N.D.U. 2008)

Example 5. Show that the roots of the equation $(1+x)^{2n} + (1-x)^{2n} = 0$ are given by

$$\pm\, i\tan\frac{(2r-1)\pi}{4n} \quad \text{where } r = 1, 2, 3, \ldots, n.$$

(P.U. 2003, 2006, 2010; G.N.D.U. 2007, 2009; Pbi. U. 2008)

Sol. The given equation is

$$(1+x)^{2n}+(1-x)^{2n}=0, \quad \text{or} \quad (1+x)^{2n}=-(1-x)^{2n}$$

or $\left(\frac{1+x}{1-x}\right)^{2n}=-1$, or $\frac{1+x}{1-x}=(-1)^{1/2n}$

$$\therefore \quad \frac{1+x}{1-x}=(\cos\pi+i\sin\pi)^{1/2n}=[\cos(2r\pi+\pi)+i\sin(2r\pi+\pi)]^{1/2n}$$

$$=[\cos(2r+1)\pi+i\sin(2r+1)\pi]^{1/2n}$$

$$=\cos\frac{(2r+1)\pi}{2n}+i\sin\frac{(2r+1)\pi}{2n}, \text{ where } r=0,1,2,\ldots\ldots,2n-1$$

$$\therefore \quad \frac{1+x}{1-x}=\frac{\cos\theta+i\sin\theta}{1}, \text{ where } \theta=\frac{(2r+1)\pi}{2n}$$

Applying componendo and dividendo, we get

$$\frac{(1+x)+(1-x)}{(1+x)-(1-x)}=\frac{(\cos\theta+i\sin\theta)+1}{(\cos\theta+i\sin\theta)-1}$$

$$\therefore \quad \frac{2}{2x}=\frac{(1+\cos\theta)+i\sin\theta}{-(1-\cos\theta)+i\sin\theta}=\frac{2\cos^2\frac{\theta}{2}+i\,.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-2\sin^2\frac{\theta}{2}+i\,.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\therefore \quad \frac{1}{x}=\frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}-\frac{1}{i}\sin\frac{\theta}{2}\right)}=\frac{1}{i}\cot\frac{\theta}{2} \quad \Rightarrow \quad x=i\tan\frac{\theta}{2}$$

or $\quad x=i\tan\frac{(2r+1)\pi}{4n}$, where $r=0,1,2,3,\ldots,2n-1$

or $\quad x=\pm i\tan\frac{(2r-1)\pi}{4n}$ where $r=1,2,3,\ldots\ldots,n.$

EXERCISE 13 (c)

1. Solve

(i) $x^3+8=0$ $\qquad$ (ii) $x^7+1=0$

2. Solve

(i) $x^4+x^3+x^2+x+1=0$ $\qquad$ (G.N.D.U. 2008)

(ii) $x^6+x^5+x^4+x^3+x^2+x+1=0$

Art-8. Important Results to be Remembered

(1) Let $x = \cos\theta + i\sin\theta$

$$\therefore \quad \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$$

$$\therefore \quad \left.\begin{aligned} x + \frac{1}{x} &= 2\cos\theta \\ \text{and} \quad x - \frac{1}{x} &= 2i\sin\theta \end{aligned}\right\} \quad \ldots(1)$$

Again $\quad x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

$$\frac{1}{x^m} = (\cos\theta - i\sin\theta)^m = \cos m\theta - i\sin m\theta$$

$$\therefore \quad x^m + \frac{1}{x^m} = 2\cos m\theta$$

$$\text{and} \quad x^m - \frac{1}{x^m} = 2i\sin m\theta \qquad \text{...(II)}$$

Results (I) and (II) will be frequently used.

(2) If n is a positive integer, then

$$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_n\, a^n$$

Students know that

(*i*) number of terms $= n + 1$

(*ii*) coefficient of terms are ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ..., ${}^nC_n$ which are called binomial coefficients.

There is a very familiar method of writing these binomial coefficients. The method is known as **Pascal Rule,** which we explain below :

Suppose we want to write the coefficients of various terms in the expansion of $\left(x + \frac{1}{x}\right)^8$. We do like this :

Index	Binomial Coefficients								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

$$\therefore \quad \left(x + \frac{1}{x}\right)^8 = x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + \frac{56}{x^2} + \frac{28}{x^4} + \frac{8}{x^6} + \frac{1}{x^8}.$$

It must be noted that powers start with 8 and go on diminishing by 2.

(*iii*) Coefficients of terms equidistant from the beginning and end are equal. It should be remembered that ${}^nC_r = {}^nC_{n-r}$.

(iv) If n is even, then there is only one middle term $T_{\frac{n}{2}+1}$.

If n is odd, then there are two middle terms $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$.

Art-9. Expansion of $\cos^n\theta$ in Terms of cosines of Multiples of θ

Let $x = \cos\theta + i\sin\theta$, $\therefore$ $x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$, $\therefore$ $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore$ $x + \frac{1}{x} = 2\cos\theta$, $x^n + \frac{1}{x^n} = 2\cos n\theta$

Now $(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$

$$= {}^nC_0 x^n + {}^nC_1 x^{n-1}.\frac{1}{x} + {}^nC_2 x^{n-2}.\frac{1}{x^2} + \dots + {}^nC_{n-2} x^2.\frac{1}{x^{n-2}} + {}^nC_{n-1} x.\frac{1}{x^{n-1}} + {}^nC_n.\frac{1}{x^n}$$

$$= {}^nC_0 x^n + {}^nC_1 x^{n-2} + {}^nC_2 x^{n-4} + \dots + {}^nC_2 \frac{1}{x^{n-4}} + {}^nC_1.\frac{1}{x^{n-2}} + {}^nC_0.\frac{1}{x^n}$$

Two cases arise :

Case I. n is even

In this case, there is only one middle term

$$T_{\frac{n}{2}+1} = {}^nC_{n/2} x^{n-\frac{n}{2}} \frac{1}{x^{n/2}} = {}^nC_{n/2}$$

$$\therefore\ (2\cos\theta)^n = {}^nC_0\left(x^n + \frac{1}{x^n}\right) + {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \dots + {}^nC_{n/2}$$

$$\therefore\ 2^n\cos^n\theta = {}^nC_0.(2\cos n\theta) + {}^nC_1\{2\cos(n-2)\theta\} + {}^nC_2\{2\cos(n-4)\theta\} + \dots + {}^nC_{n/2}$$

$$\therefore\ \cos^n\theta = \frac{1}{2^{n-1}}\left[{}^nC_0\cos n\theta + {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta + \dots + \frac{1}{2}.{}^nC_{n/2}\right]$$

Case II. n is odd

In this case, there are two middle terms $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$.

$$T_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}} x^{n-\frac{n-1}{2}} \cdot \frac{1}{x^{\frac{n-1}{2}}} = {}^nC_{\frac{n-1}{2}} x$$

$$T_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}} x^{n-\frac{n+1}{2}} \cdot \frac{1}{x^{\frac{n+1}{2}}} = {}^nC_{\frac{n-1}{2}} \cdot \frac{1}{x}$$

$$\therefore\ 2(\cos\theta)^n = {}^nC_0\left(x^n + \frac{1}{x^n}\right) + {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

$$+ {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \ldots + {}^nC_{\frac{n-1}{2}}\left(x + \frac{1}{x}\right)$$

$$\therefore\ 2^n \cos^n\theta = {}^nC_0 . 2\cos n\theta + {}^nC_1 . 2\cos(n-2)\theta$$

$$+ {}^nC_2 . 2\cos(n-4)\theta + \ldots + {}^nC_{\frac{n-1}{2}} . 2\cos\theta$$

$$\therefore\ \cos^n\theta = \frac{1}{2^{n-1}}\left[{}^nC_0 \cos n\theta + {}^nC_1 \cos(n-2)\theta\right.$$

$$\left. + {}^nC_2 \cos(n-4)\theta + \ldots + {}^nC_{\frac{n-1}{2}} \cos\theta\right]$$

Note : Students can find themselves the expansion of $\sin^n\theta$ in terms of cosines or sines of multiples of θ according as n is even or odd integer. They should use the formula $(2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$.

ILLUSTRATIVE EXAMPLES

Example 1. Prove that $\cos^7\theta = \frac{1}{2^6}[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos$ ...

(Pbi. U. 2009; H.P.U. 2012)

Sol. Let $x = \cos\theta + i\sin\theta$

$\therefore\ x^m = \cos m\theta - i\sin m\theta$

and $\quad \dfrac{1}{x} = \cos\theta - i\sin\theta,$

$\therefore \quad \dfrac{1}{x^m} = \cos m\theta - i\sin m\theta$

Now $\quad 2\cos\theta = x + \dfrac{1}{x},\ 2\cos m\theta = x^m + \dfrac{1}{x^m}$

$\therefore \quad (2\cos\theta)^7 = \left(x + \dfrac{1}{x}\right)^7$

Now by Pascal's Rule,

1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

$$\therefore \quad (2\cos\theta)^7 = x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

$$\therefore \quad 2^7 . \cos^7\theta = 2\cos 7\theta + 7 . 2\cos 5\theta + 21 . 2\cos 3\theta + 35 . 2\cos\theta$$

$$\Rightarrow \quad \cos^7\theta = \frac{1}{2^6}\left[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right].$$

Example 2. Prove that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(H.P.U. 2013)

Sol. Let $\quad x = \cos\theta + i\sin\theta, \quad \therefore \quad x^m = \cos m\theta + i\sin m\theta$

and $\quad \dfrac{1}{x} = \cos\theta - i\sin\theta, \quad \therefore \quad \dfrac{1}{x^m} = \cos m\theta - i\sin m\theta$

$\therefore \quad 2i\sin\theta = x - \dfrac{1}{x},\ 2i\sin m\theta = x^m - \dfrac{1}{x^m}$

$$\therefore \quad (2i\sin\theta)^5 = \left(x - \frac{1}{x}\right)^5 = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5} \quad \text{[By Pascal's Rule]}$$

$$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$\therefore \quad 32\, i \sin^5\theta = 2\, i \sin 5\theta - 5\cdot 2\, i \sin 3\theta + 10\cdot 2\, i \sin\theta$

or $\quad 16 \sin^5\theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin\theta.$

Example 3. Prove that

$$\cos^6\theta \sin^4\theta = 2^{-9}\,[\cos 10\theta + 2\cos 8\theta - 3\cos 6\theta - 8\cos 4\theta + 2\cos 2\theta + 6]$$

(H.P.U. 200...)

Sol. Let $x = \cos\theta + i\sin\theta, \quad \therefore \quad \dfrac{1}{x} = \cos\theta - i\sin\theta$

$$\Rightarrow \quad 2\cos\theta = x + \frac{1}{x}, \quad 2\, i \sin\theta = x - \frac{1}{x}$$

$$\therefore \quad (2\cos\theta)^6\,(2\, i\sin\theta)^4 = \left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^4.$$

Now to obtain the coefficients of the various powers of x in the product, we first write the coefficients in the expansion of $\left(x+\dfrac{1}{x}\right)^6$, from Pascal's Table, and then multiply by $x-\dfrac{1}{x}$ four times in succession.

	1	6	15	20	15	6	1				
	1	−1									
	1	6	15	20	15	6	1				
		−1	−6	−15	−20	−15	−6	−1			
I	1	5	9	5	−5	−9	−5	−1			
	1	−1									
	1	5	9	5	−5	−9	−5	−1			
		−1	−5	−9	−5	5	9	5	1		
II	1	4	4	−4	−10	−4	4	4	1		
	1	−1									
	1	4	4	−4	−10	−4	4	4	1		
		−1	−4	−4	4	10	4	−4	−4	−1	
III	1	3	0	−8	−6	6	8	0	−3	−1	
	1	−1									
	1	3	0	−8	−6	6	8	0	−3	−1	
		−1	−3	0	8	6	−6	−8	0	3	1
IV	1	2	−3	−8	2	12	2	−8	−3	2	

These are the required coefficients.

Hence $\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^4$

$$= x^{10} + 2x^8 - 3x^6 - 8x^4 + 2x^2 + 12 + \frac{2}{x^2} - \frac{8}{x^4} - \frac{3}{x^6} + \frac{2}{x^8} + \frac{1}{x^{10}}$$

$$= \left(x^{10} + \frac{1}{x^{10}}\right) + 2\left(x^8 + \frac{1}{x^8}\right) - 3\left(x^6 + \frac{1}{x^6}\right) - 8\left(x^4 + \frac{1}{x^4}\right) + 2\left(x^2 + \frac{1}{x^2}\right) + 12$$

$\therefore$ $(2\cos\theta)^6 (2i\sin\theta)^4 = 2\cos 10\theta + 2(2\cos 8\theta) - 3(2\cos 6\theta)$

$- 8(2\cos 4\theta) + 2(2\cos 2\theta) + 12$

$\therefore$ $2^{10}\cos^6\theta\sin^4\theta = 2[\cos 10\theta + 2\cos 8\theta - 3\cos 6\theta - 8\cos 4\theta + 2\cos 2\theta + 6]$

$\therefore$ $\cos^6\theta\sin^4\theta = 2^{-9}[\cos 10\theta + 2\cos 8\theta - 3\cos 6\theta - 8\cos 4\theta + 2\cos 2\theta + 6]$

EXERCISE 13 (*d*)

1. Prove that

$$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$$

(H.P.U. 2005, 2006, 2013; Pbi. U. 2009)

2. Express $\cos^8\theta$ in a series of cosines of multiples of θ. (G.N.D.U. 2002)
3. Express $\cos^9\theta$ in a series of cosines of multiple of θ.
4. Prove that

$$\sin^6\theta = -\frac{1}{32}[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10]$$

5. Expand $\sin^7\theta$ in a series of multiples of θ. (P.U. 2010)
6. Prove that

$$\sin^8\theta = \frac{1}{128}[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35]$$

(G.N.D.U. 2012)

7. Express $\cos^4\theta\sin^3\theta$ in a series of sines of multiples of θ.

(H.P.U. 2008)

8. Express $\sin^6\theta\cos^2\theta$ in terms of cosines of multiples of θ.

ANSWERS

2. $\frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$

3. $\frac{1}{2^8}[\cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta]$

5. $-\frac{1}{64}$ [sin 7θ – 7 sin 5θ + 21 sin 3θ – 35 sin θ]

7. $-\frac{1}{64}$ [sin 7θ + sin 5θ – 3 sin 3θ – 3 sin θ]

8. $-\frac{1}{2^7}$ [cos 8θ – 4 cos 6θ + 4 cos 4θ + 4 cos 2θ – 5]

Art-10. Expand cos $n\theta$ and sin $n\theta$ in terms of the t-ratios of θ, n being a positive integer.

Proof : By De Moivre's Theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

or
$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= {}^nC_0\cos^n\theta + {}^nC_1\cos^{n-1}\theta.(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2$$

$$+ {}^nC_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots + {}^nC_{n-1}\cos\theta\,(i\sin\theta)^{n-1} + {}^nC_n(i\sin\theta)^n$$

$$= ({}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots)$$

$$+ i\,({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots)$$

Equating real and imaginary parts,

$$\cos n\theta = {}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots$$

and
$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots$$

Cor. $\therefore$ $\tan n\theta = \dfrac{\sin n\theta}{\cos n\theta}$

$$= \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\sin^3\theta + \ldots.}{{}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots..}$$

$$= \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \ldots..}{{}^nC_0 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots..}$$

[Dividing num. and den. by $\cos^n\theta$]

ILLUSTRATIVE EXAMPLES

Example 1. Expand cos 7θ and sin 7θ in powers of cos θ and sin θ. Hence, obtain an expression for tan 7θ in powers of tan θ.

(Pbi. U. 2010)

Sol. By De Moivre's Theorem,

$\cos 7\theta + i \sin 7\theta = (\cos\theta + i\sin\theta)^7$

$= {}^7C_0 \cos^7\theta + {}^7C_1 \cos^6\theta\,(i\sin\theta) + {}^7C_2 \cos^5\theta\,(i\sin\theta)^2$
$+ {}^7C_3 \cos^4\theta\,(i\sin\theta)^3 + {}^7C_4 \cos^3\theta\,(i\sin\theta)^4 + {}^7C_5 \cos^2\theta\,(i\sin\theta)^5$
$+ {}^7C_6 \cos\theta\,(i\sin\theta)^6 + {}^7C_7 (i\sin\theta)^7$

$= (\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta)$
$+ i(7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta)$

Equating real and imaginary parts,

$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta,$

$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

$$\therefore \tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta} = \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta}$$

$$= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

EXERCISE 13 (*e*)

1. Express $\sin 5\theta$ in terms of sines and cosines of an angle θ.

(Pbi. U. 2008; P.U. 2011)

2. Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

3. Expand $\sin 6\theta$ is powers of $\sin\theta$ and $\cos\theta$ and deduce that

$$\frac{\sin 6\theta}{\sin\theta} = 6\cos\theta - 32\cos^3\theta + 32\cos^5\theta.$$

(G.N.D.U. 2003; Pbi. U. 2010)

ANSWERS

1. $5\cos^4\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Art-11. Expansion of $\tan(\theta_1 + \theta_2 + \ldots + \theta_n)$

Prove that $\tan(\theta_1 + \theta_2 + \ldots + \theta_n) = \dfrac{S_1 - S_3 + S_5 - \ldots..}{1 - S_2 + S_4 - \ldots..}$

where S_r stands for the sum of the products of $\tan\theta_1, \tan\theta_2, \ldots, \tan\theta_n$ taken r at a time.

Proof : We have

$\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$ ...(1)

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$

Now $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1\left(1 + i\dfrac{\sin\theta_1}{\cos\theta_1}\right) = \cos\theta_1(1 + i\tan\theta_1)$

$\cos\theta_2 + i\sin\theta_2 \qquad = \cos\theta_2(1 + i\tan\theta_2)$

....

$\cos\theta_n + i\sin\theta_n \qquad = \cos\theta_n(1 + i\tan\theta_n)$

$\therefore$ from (1), we get,

$$\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n\,.\,(1 + i\tan\theta_1)(1 + i\tan\theta_2)\ldots(1 + i\tan\theta_n)$$

$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n\,[1 + iS_1 + i^2S_2 + i^3S_3 + \ldots + i^nS_n]$$

$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n\,[(1 - S_2 + S_4 - \ldots) + i(S_1 - S_3 + S_5 - \ldots)]$$

Equating real and imaginary parts,

$$\cos(\theta_1 + \theta_2 + \ldots + \theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1 - S_2 + S_4 - \ldots) \quad \ldots(2)$$

$$\sin(\theta_1 + \theta_2 + \ldots + \theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(S_1 - S_3 + S_5 - \ldots) \quad \ldots(3)$$

Dividing (3) by (2), we get,

$$\tan(\theta_1 + \theta_2 + \ldots + \theta_n) = \frac{S_1 - S_3 + S_5 - \ldots..}{1 - S_2 + S_4 - \ldots..}$$

ILLUSTRATIVE EXAMPLES

Example 1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + p = 0$, prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$, except in one particular case.

(H.P.U. 2009)

Sol. Since α, β, γ are roots of the equation $x^3 + px^2 + qx + p = 0$

$\therefore \quad \Sigma\alpha = \alpha + \beta + \gamma = -p$

$\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -p$

Let $\quad \alpha = \tan\theta_1,\ \beta = \tan\theta_2,\ \gamma = \tan\theta_3$

$\therefore \quad \theta_1 = \tan^{-1}\alpha,\ \theta_2 = \tan^{-1}\beta,\ \theta_3 = \tan^{-1}\gamma$

Now $S_1 = \Sigma\alpha = \Sigma\tan\theta_1 = -p$

$S_2 = \Sigma\alpha\beta = \Sigma\tan\theta_1\tan\theta_2 = q$

$S_3 = \alpha\beta\gamma = \tan\theta_1\tan\theta_2\tan\theta_3 = -p$

Also, $\tan(\theta_1+\theta_2+\theta_3)=\dfrac{S_1-S_3}{1-S_2}=\dfrac{-p+p}{1-q}=\dfrac{0}{1-q}=0$, provided $q\neq 1$

$\therefore \quad \theta_1+\theta_2+\theta_3=n\pi$ where $n\in I$ and $q\neq 1$.

$\therefore \quad \tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=n\pi$, except when $q=1$.

Example 2. Prove that the equation $a\,h\sec\theta-b\,k\,\text{cosec}\,\theta=a^2-b^2$ has four roots and that the sum of the four values of θ which satisfy it is equal to an odd multiple of π radians.

Sol. The given equation is

$$a\,h\sec\theta-b\,k\,\text{cosec}\,\theta-(a^2-b^2)=0 \qquad \ldots(1)$$

Let $t=\tan\dfrac{\theta}{2}$

$$\therefore \quad \sec\theta=\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\frac{1+t^2}{1-t^2},\quad \text{cosec}\,\theta=\frac{1+\tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}=\frac{1+t^2}{2t}$$

$\therefore$ given equation (1) becomes

$$a\,h\left(\frac{1+t^2}{1-t^2}\right)-b\,k\left(\frac{1+t^2}{2t}\right)-(a^2-b^2)=0$$

or $\quad 2\,a\,h\,t\,(1+t^2)-b\,k\,(1-t^2)(1+t^2)-2\,t\,(a^2-b^2)(1-t^2)=0$

or $\quad 2\,a\,h\,t+2\,a\,h\,t^3-b\,k+b\,k\,t^4-2\,(a^2-b^2)\,t+2\,(a^2-b^2)\,t^3=0$

or $\quad b\,k\,t^4+(2\,a\,h+2\,a^2-2\,b^2)\,t^3+(2\,a\,h-2\,a^2+2\,b^2)\,t-b\,k=0$

Let its roots be $\tan\dfrac{\theta_1}{2},\ \tan\dfrac{\theta_2}{2},\ \tan\dfrac{\theta_3}{2},\ \tan\dfrac{\theta_4}{2}$

$$\therefore \quad S_1=-\frac{2\,a\,h+2\,a^2-2\,b^2}{b\,k},\quad S_2=0$$

$$S_3=-\frac{2\,a\,h-2\,a^2+2\,b^2}{b\,k},\quad S_4=-\frac{b\,k}{b\,k}=-1$$

Now $\cot\left(\dfrac{\theta_1}{2}+\dfrac{\theta_2}{2}+\dfrac{\theta_3}{2}+\dfrac{\theta_4}{2}\right)=\dfrac{1-S_2+S_4}{S_1-S_3}$

$$=\frac{1-0-1}{\frac{2\,a\,h-2\,a^2-2\,b^2}{b\,k}+\frac{2\,a\,h-2\,a^2+2\,b^2}{b\,k}}=0$$

$\therefore \quad \cot\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)=\cot\frac{\pi}{2}$

$\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}=n\pi+\frac{\pi}{2}$

$\therefore \quad \theta_1+\theta_2+\theta_3+\theta_4=(2n+1)\pi$, which is an odd multiple of π.

EXERCISE 13 (f)

1. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation
$x^4 - x^3\sin 2\theta + x^2\cos 2\theta - x\cos\theta - \sin\theta = 0,$
prove that $\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma+\tan^{-1}\delta = n\pi+\frac{\pi}{2}-\theta.$

2. If $\theta_1, \theta_2, \theta_3$ be the three values of θ which satisfy the equation $\tan 2\theta = \lambda\tan(\theta+\alpha)$ and be such that no two of these differ by a multiple of π, prove that $\theta_1+\theta_2+\theta_3+\alpha$ is multiple of π.

3. Prove that the equation $a^2\cos^2\theta + b^2\sin^2\theta + 2ga\cos\theta + 2fb\sin\theta + c = 0$ has four roots and that the sum of the values which satisfy it is an even multiple of π radius.

OR

Prove that the sum of the eccentric angles of the points of intersection of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ and the circle $x^2+y^2+2gx+2fy+c=0$ is an even multiple of π radians.

4. Prove that the equation $\sin 3\theta = a\sin\theta + b\cos\theta + c$ has six roots and the sum of the six values of θ which satisfy it is equal to an odd multiple of π radian.

(G.N.D.U. 2011, 2012)

Art-12. Formation of Equation

We explain the method by examples given below :

ILLUSTRATIVE EXAMPLES

Example 1. (*a*) Prove that $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{8\pi}{7}$ are the roots of

$8x^3+4x^2-4x-1=0.$

Hence form an equation whose roots are $\sec\frac{2\pi}{7}, \sec\frac{4\pi}{7}, \sec\frac{6\pi}{7}$.

(b) Also form an equation whose roots are $\sec^2\frac{2\pi}{7}, \sec^2\frac{4\pi}{7}, \sec^2\frac{6\pi}{7}$ and prove that $\tan^2\frac{2\pi}{7} + \tan^2\frac{4\pi}{7} + \tan^2\frac{6\pi}{7} = 21$.

Sol. Let $\theta = \frac{2n\pi}{7}$

where n is any integer, zero, positive or negative.

By giving to n the values 0, 1, 2, 4, the values of $\cos\theta$ are

$$1, \cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{8\pi}{7}$$

For $n = 3$, $\cos\theta = \cos\frac{6\pi}{7} = \cos\left(2\pi - \frac{8\pi}{7}\right) = \cos\frac{8\pi}{7}$

For $n = 5$, $\cos\theta = \cos\frac{10\pi}{7} = \cos\left(2\pi - \frac{4\pi}{7}\right) = \cos\frac{4\pi}{7}$

For $n = 6$, $\cos\theta = \cos\frac{12\pi}{7} = \cos\left(2\pi - \frac{2\pi}{7}\right) = \cos\frac{2\pi}{7}$

and so on.

$\therefore$ no new value of $\cos\theta$ is obtained by giving to n, the values 3, 5, 6, 7, ...

Now $7\theta = 2n\pi \Rightarrow 4\theta + 3\theta = 2n\pi$

$\therefore$ $4\theta = 2n\pi - 3\theta \Rightarrow \cos 4\theta = \cos(2n\pi - 3\theta)$

$\Rightarrow$ $\cos 4\theta = \cos 3\theta \Rightarrow 2\cos^2 2\theta - 1 = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow$ $2(2\cos^2\theta - 1)^2 - 1 = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow$ $2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 - (4\cos^3\theta - 3\cos\theta) = 0$

$\Rightarrow$ $8\cos^4\theta - 4\cos^3\theta - 8\cos^2\theta + 3\cos\theta + 1 = 0$

Putting $x = \cos\theta$, we get,

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0 \qquad \text{...(1)}$$

Put $x = 1$ in (1).

$\therefore$ $8 - 4 - 8 + 3 + 1 = 0$

or $0 = 0$

$\Rightarrow$ $x = 1$ is root of (1)

$\Rightarrow$ $x - 1$ is factor of L.H.S. of (1).

1	8	−4	−8	3	1
		8	4	−4	−1
	8	4	−4	−1	0

$\therefore$ remaining roots of (1) are given by

$$8x^3 + 4x^2 - 4x - 1 = 0$$

Now $x = 1 \Rightarrow \cos\theta = 1$ which refers to $\cos\dfrac{2n\pi}{7}$ when $n = 0$.

$\therefore$ the remaining roots $\cos\dfrac{2\pi}{7}$, $\cos\dfrac{4\pi}{7}$, $\cos\dfrac{8\pi}{7}$

are the roots of equation (2)

i.e., $8x^3 + 4x^2 - 4x - 1 = 0$

Now we are to find an equation whose roots are

$$\sec\frac{2\pi}{7},\ \sec\frac{4\pi}{7},\ \sec\frac{6\pi}{7}$$

i.e., $\sec\dfrac{2\pi}{7},\ \sec\dfrac{4\pi}{7},\ \sec\dfrac{8\pi}{7}$ $\left[\because \sec\dfrac{6\pi}{7} = \sec\left(2\pi - \dfrac{8\pi}{7}\right) = \sec\dfrac{8\pi}{7}\right.$

Let new equation be in y.

$$\therefore \quad y = \frac{1}{x} \quad \text{or} \quad x = \frac{1}{y}$$

Putting $x = \dfrac{1}{y}$ in (1), we get,

$$\frac{8}{y^3} + \frac{4}{y^2} - \frac{4}{y} - 1 = 0$$

or $\quad 8 + 4y - 4y^2 - y^3 = 0$

or $\quad y^3 + 4y^2 - 4y - 8 = 0$ ...(3)

Its roots are $\sec\dfrac{2\pi}{7},\ \sec\dfrac{4\pi}{7},\ \sec\dfrac{6\pi}{7}$

$$\therefore \quad \sec\frac{2\pi}{7} + \sec\frac{4\pi}{7} + \sec\frac{6\pi}{7} = -4$$

(*b*) We want to form an equation whose roots are

$$\sec^2\frac{2\pi}{7},\ \sec^2\frac{4\pi}{7},\ \sec^2\frac{6\pi}{7}.$$

Let new equation be in t.

$$\therefore \quad t = y^2 \text{ or } y = \sqrt{t}$$

Putting $y = \sqrt{t}$ in (3), we get,

$$t\sqrt{t} + 4t - 4\sqrt{t} - 8 = 0$$

or $\quad (t-4)\sqrt{t} = 8 - 4t$

or $\quad t(t-4)^2 = (8-4t)^2$

or $\quad t(t^2 - 8t + 16) = 64 + 16t^2 - 64t$

or $\quad t^3 - 24t^2 + 80t - 64 = 0$

Its roots are $\sec^2\frac{2\pi}{7}, \sec^2\frac{4\pi}{7}, \sec^2\frac{6\pi}{7}$ ...(4)

$$\therefore \quad \sec^2\frac{2\pi}{7} + \sec^2\frac{4\pi}{7} + \sec^2\frac{6\pi}{7} = 24$$

$$\Rightarrow \quad 1+\tan^2\frac{2\pi}{7} + 1 + \tan^2\frac{4\pi}{7} + 1 + \tan^2\frac{6\pi}{7} = 24$$

$$\Rightarrow \quad \tan^2\frac{2\pi}{7} + \tan^2\frac{4\pi}{7} + \tan^2\frac{6\pi}{7} = 21.$$

EXERCISE 13 (g)

1. (*a*) Form an equation whose roots are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$.

Hence evaluate $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7}$.

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(*b*) Hence obtain the equations whose roots are

(*i*) $\sec^2\frac{\pi}{7}, \sec^2\frac{3\pi}{7}, \sec^2\frac{5\pi}{7}$

(*ii*) $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$

(*iii*) Deduce that $\cot^2\frac{\pi}{7} . \cot^2\frac{3\pi}{7} . \cot^2\frac{5\pi}{7} = \frac{1}{7}$.

2. Prove that the roots of the equation $x^4 + 2x^3 - x^2 - 2x + 1 = 0$ are $2\cos\frac{2r\pi}{5}$, where $r = 1, 2, 3, 4$.

3. Prove that the four roots of the equation $16x^4 - 20x^2 + 5 = 0$ are

$$\pm \sin\frac{\pi}{5}, \ \pm \sin\frac{2\pi}{5}.$$

4. Show that $\tan\frac{\theta}{7} + \tan\frac{\theta+\pi}{7} + \tan\frac{\theta+2\pi}{7} + \ldots + \tan\frac{\theta+6\pi}{7} = 7\tan\theta$.

ANSWERS

1. (*a*) $8x^3 - 4x^2 - 4x + 1 = 0; 4$

(*b*) (*i*) $t^3 - 24t^2 + 80t - 64 = 0$ (*ii*) $z^3 - 21z^2 + 35z - 7 = 0$