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# SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS

**Art-1.** Explain Cardan's method of solving the cubic $a_0 x^3 + 3 a_1 x^2 + 3 a_2 x + a_3 = 0$.

(P.U. 2000, 2002 ; G.N.D.U. 2001, 2004, 2006)

**Proof :** The given equation is

$$a_0 x^3 + 3 a_1 x^2 + 3 a_2 x + a_3 = 0 \quad ...(1)$$

By the transformation $z = a_0 x + a_1$, equation (1) reduces to

$$z^3 + 3Hz + G = 0 \quad ...(2)$$

where $H = a_0 a_2 - a_1^2, G = a_0^2 a_3 - 3 a_0 a_1 a_2 + 2 a_1^3$

Let $z = u + v$

Cubing both sides, we get,

$$z^3 = u^3 + v^3 + 3 u v (u + v)$$

or $z^3 = u^3 + v^3 + 3 u v z$ [$\because$ of (3)]

or $z^3 - 3 u v z - (u^3 + v^3) = 0.$

Comparing the coefficients of (2) and (4), we get,

$-u v = H$ or $u v = -H$

$\therefore \quad u^3 v^3 = -H^3$

and $-(u^3 + v^3) = G$ or $u^3 + v^3 = -G$

$\therefore \quad u^3$ and $v^3$ are the roots of the equation $t^2 + G t - H^3 = 0$

$$\therefore \quad t = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2}$$

$$\therefore \quad u^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2}, \quad v^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2}$$

The three cube roots of $u^3$ are $u, u\omega, u\omega^2$ and those of $v^3$ are $v, v\omega, v\omega^2$ where $\omega$ is a cube root of unity.

Now $z = u + v$ such that $u v = -H$, where H is real.

Therefore to find $z$, we should add a cube root of $u^3$ and a cube root of $v^3$ in such a manner that their product is real.

If three values of $u$ are $u, u\,\omega, u\,\omega^2$, then the corresponding values of $v$ are

$$\frac{-H}{u}, \frac{-H}{u\omega}, \frac{-H}{u\omega^2} \quad i.e., \quad \frac{-H}{u}, \frac{-H\,\omega^2}{u}, \frac{-H\,\omega}{u} \qquad [\because \omega^3 = 1]$$

$$\therefore \quad z = u + \left(\frac{-H}{u}\right), u\,\omega + \left(\frac{-H\,\omega^2}{u}\right), u\,\omega^2 + \left(\frac{-H\,\omega}{u}\right).$$

Putting these values of $z$, one by one, in $z = a_0 x + a_1$ we get three values of $x$, which are roots of given equation (1).

## Art-2. Discriminant of an Equation

The discriminant of the equation $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$, is defined as $a_0^{2(n-1)} \prod (\alpha_j - \alpha_k)^2$, $1 \le j < k \le n$ where $\alpha_j$ $(1 \le j \le n)$ are the roots of the given equation.

For example, let $\alpha, \beta$ be roots of $a x^2 + b x + c = 0$

$$\therefore \quad \text{disc.} = a^{2(2-1)} (\alpha - \beta)^2 = a^2 [(\alpha + \beta)^2 - 4\,\alpha\beta]$$

$$= a^2 \left[\left(-\frac{b}{a}\right)^2 - \frac{4c}{a}\right] = a^2 \left(\frac{b^2}{a^2} - \frac{4c}{a}\right) = b^2 - 4\,a\,c$$

$\therefore$ discriminant of $a_0 x^3 + 3\,a_1 x^2 + 3\,a_2 x + a_3 = 0$ is

$$a_0^{2(3-1)} (\alpha_1 - \alpha_2)^2 (\alpha_1 - \alpha_3)^2 (\alpha_2 - \alpha_3)^2$$

where $\alpha_1, \alpha_2, \alpha_3$ are roots of the equation.

**Art-3.** Discuss the nature of roots of the cubic $z^3 + 3Hz + G = 0$. (Pbi. U. 2010)

**Proof:** We have

$$z = u + \left(\frac{-H}{u}\right), u\,\omega + \left(\frac{-H}{u}\omega^2\right), u\,\omega^2 + \left(\frac{-H}{u}\omega\right).$$

(Reproduce Art-1 up to this)

or $\quad z = u + v, u\,\omega + v\,\omega^2, u\,\omega$ where $v = \dfrac{-H}{u}$.

**Case I.** When $G^2 + 4H^3 = 0$ and $G \neq 0$, then $u^3 = v^3$ *i.e.*, $u = v$.

$\therefore$ roots of the given equation become

$2u, u(\omega + \omega^2), u(\omega^2 + \omega)$ *i.e.*, $2u, -u, -u$

$\therefore$ two roots are equal.

**Case II.** When $G^2 + 4H^3 = 0$ and $G = 0$ and hence $H = 0$, the given equation becomes $z^3 = 0$.

$\therefore \quad z = 0, 0, 0$

$\therefore$ all the three roots are equal.

**Case III.** When $G^2 + 4H^3 > 0$, then $u^3$ and $v^3$ are both real. Let $u, v$ be both real

$\therefore$ roots of given equation are $u + v,\ u\omega^2 + v\omega,\ u\omega + v + \omega^2$.

$\therefore$ one root $u + v$ is real and other two roots are complex roots.

**Case IV.** When $G^2 + 4H^2 < 0$, then $u^3$ and $v^3$ are complex conjugates and hence $u$ and $v$ are complex conjugates.

Let $u = a + ib,\ v = a - ib$

$$\therefore \quad u + v = a + ib + a - ib = 2a$$

$$u\omega + v\omega^2 = (a + ib)\,\omega + (a - ib)\,\omega^2 = a(\omega + \omega^2) + ib(\omega - \omega^2)$$

$$= a(-1) + ib\left[\frac{-1 + i\sqrt{3}}{2} - \frac{-1 - i\sqrt{3}}{2}\right] = -a + ib\left(i\sqrt{3}\right) = -a - b\sqrt{3}$$

$$u\omega^2 + v\omega = (a + ib)\,\omega^2 + (a - ib)\,\omega = a(\omega^2 + \omega) + ib(\omega^2 - \omega)$$

$$= -a + b\sqrt{3}$$

(as above)

$\therefore$ all the three roots are real and distinct.

**Note :** In the case when $G^2 + 4H^3 < 0$, then $u^3$ and $v^3$ are both imaginary. There is no algebraic method of finding out cube roots of an imaginary quantity. For this reason, this case of Cardan's solution is called **Irreducible case of Cardan's Solution.** We use D'Moivre's Theorem to find the solution in this case.

## Art-4. Discriminant of the Cubic $z^3 + Hz + G = 0$.

(H.P.U. 2001; G.N.D.U. 200...)

We know that the roots of the equation $z^3 + 3Hz + G = 0$

are $u + v,\ uw + vw^2$ and $uw^2 + vw$ where $u^3, v^3$ are given by

$$u^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2}, \quad v^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2}$$

Let $\alpha_1, \alpha_2, \alpha_3$ be roots of (1)

$$\therefore \quad \alpha_1 = u + v,\ \alpha_2 = uw + vw^2,\ \alpha_3 = uw^2 + vw$$

Now $\alpha_1 - \alpha_2 = (u + v) - (uw + vw^2) = u + v - uw - vw^2$

$$= u(1 - w) + v(1 - w^2) = (1 - w)\left[u + v(1 + w)\right]$$

$$= (1 - w)\left[u + v(-w^2)\right] \text{ as } 1 + w + w^2 = 0$$

$$= (1 - w)(u - vw^2)$$

$$\alpha_1 - \alpha_3 = (u + v) - (uw^2 + vw) = u + v - uw^2 - vw = u(1 - w^2) + v(1 - \ldots$$

$$= (1 - w)\left[u(1 + w) + v\right] = (1 - w)(-uw^2 + v)$$

$$= (1 - w)\left[-uw^2 + vw^3\right] = w^2(1 - w)(-u + vw)$$

$\alpha_2 - \alpha_3 = (u w + v w^2) - (u w^2 + v w)$

$= u w + v w^2 - u w^2 - v w = u w (1 - w) - v w (1 - w)$

$= w (1 - w) (u - v)$

$\therefore \ (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)$

$= (1 - w)(u - v w^2) \cdot w^2 (1 - w)(- u + v w) \cdot w (1 - w)(u - v)$

$= w^3 (1 - w)^3 (u - v w^2)(- u + v w)(u - v)$

$= (1)(1 - 3 w + 3 w^2 - w^3)\left[- u^2 + u v w + u v w^2 - v^2 w^2\right](u - v)$

$= (1 - 3 w + 3 w^2 - 1)\left[- u^2 + u v (w + w^2) - v^2\right](u - v)$

$= (- 3 w + 3 w^2)(u - v)(u^2 + u v + v^2)$

$$= 3\left[\frac{-1 + i\sqrt{3}}{2} - \frac{-1 - i\sqrt{3}}{2}\right](u^3 - v^3)$$

$$= 3\left(i\sqrt{3}\right)\left[\frac{-G + \sqrt{G^2 + 4H^3}}{2} - \frac{-G - \sqrt{G^2 + 4H^3}}{2}\right]$$

$\therefore \ (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) = i3\sqrt{3}\sqrt{G^2 + 4H^3}$ ...(2)

Disc. of (1) $= (1)^{2(3-1)} (\alpha_1 - \alpha_2)^2 (\alpha_1 - \alpha_3)^2 (\alpha_2 - \alpha_3)^2$ $[\because \ a_0 = 1]$

$= (1)^4 \left[i 3\sqrt{3}\sqrt{G^2 + 4H^3}\right]^2$ $[\because$ of (2)]

$= - 27 (G^2 + 4H^3)$

**Cor.** Discriminant of $a_0 x^3 + 3 a_1 x^2 + 3 a_2 x + a_3 = 0.$ (Pbi. U. 2010)

**Proof :** The given equation is

$a_0 x^3 + 3 a_1 x^2 + 3 a_2 x + a_3 = 0$

Put $z = a_0 x + a_1$

$\therefore$ transformed equation is

$z^3 + 3Hz + G = 0$ ...(2)

where $H = a_0 a_2 - a_1^{\ 2}$, $G = a_0^{\ 2} a_3 - 3 a_0 a_1 a_2 + 2 a_1^{\ 3}$

If $x_1, x_2, x_3$ are roots of (1) and $\alpha_1, \alpha_2, \alpha_3$ are roots of (2), then

$\alpha_1 = a_0 x_1 + a_1$, $\alpha_2 = a_0 x_2 + a_1$, $\alpha_3 = a_0 x_3 + a_1$

$\therefore \ \alpha_1 - \alpha_2 = a_0 (x_1 - x_2)$, $\alpha_1 - \alpha_3 = a_0 (x_1 - x_3)$, $\alpha_2 - \alpha_3 = a_0 (x_2 - x_3)$

$\therefore \ (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) = a_0^{\ 3} (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

$$\text{Disc. of (1)} = a_0^{2(3-1)} (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2$$

$$= a_0^4 (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2$$

$$= a_0^4 \cdot \frac{1}{a_0^6} (\alpha_1 - \alpha_2)^2 (\alpha_1 - \alpha_3)^2 (\alpha_2 - \alpha_3)^2$$

$$= \frac{1}{a_0^2} [-27 (G^2 + 4H^3)]$$

$$= -\frac{27}{a_0^2} (G^2 + 4H^3).$$

**Art-5. Trigonometric Solution of a Real Cubic with Real Roots**

Use trigonometric method to solve the cubic

$$x^3 + 3Hx + G = 0 \text{ when } G^2 + 4H^3 < 0, G, G \in \mathbf{R}.$$

**Proof :** We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Replacing $\theta$ by $\theta + \frac{2\pi}{3}$ in (1), we get,

$$\cos 3\left(\theta + \frac{2\pi}{3}\right) = 4\cos^3\left(\theta + \frac{2\pi}{3}\right) - 3\cos\left(\theta + \frac{2\pi}{3}\right)$$

$$\therefore \quad \cos(3\theta + 2\pi) = 4\cos^3\left(\theta + \frac{2\pi}{3}\right) - 3\cos\left(\theta + \frac{2\pi}{3}\right)$$

$$\Rightarrow \quad \cos 3\theta = 4\cos^3\left(\theta + \frac{2\pi}{3}\right) - 3\cos\left(\theta + \frac{2\pi}{3}\right)$$

Again replacing $\theta$ by $\theta + \frac{4\pi}{3}$ in (1), we get,

$$\cos 3\left(\theta + \frac{4\pi}{3}\right) = 4\cos^3\left(\theta + \frac{4\pi}{3}\right) - 3\cos\left(\theta + \frac{4\pi}{3}\right)$$

or $$\cos(3\theta + 4\pi) = 4\cos^3\left(\theta + \frac{4\pi}{3}\right) - 3\cos\left(\theta + \frac{4\pi}{3}\right)$$

or $$\cos 3\theta = 4\cos^3\left(\theta + \frac{4\pi}{3}\right) - 3\cos\left(\theta + \frac{4\pi}{3}\right)$$

From (1), (2) and (3), it is clear that $\cos\theta$, $\cos\left(\theta + \frac{\pi}{3}\right)$

and $\cos\left(\theta+\frac{4\pi}{3}\right)$ are roots of the equation

$$\cos 3\theta = 4z^3 - 3z$$

or $$z^3 - \frac{3}{4}z = \frac{1}{4}\cos 3\theta \qquad \text{...(4)}$$

The given equation is

$$x^3 + 3Hx + G = 0$$

Putting $x = rz$ where $r$ is to be chosen properly, we get,

$$(rz)^3 + 3H(rz) + G = 0$$

or $$r^3z^3 + 3Hrz + G = 0$$

or $$z^3 + \frac{3H}{r^2}z + \frac{1}{r^3}G = 0$$

or $$z^3 + \frac{3H}{r^2}z = -\frac{G}{r^3} \qquad \text{...(5)}$$

Comparing (3) and (4), we get,

$$\frac{3H}{r^2} = -\frac{3}{4} \qquad \text{...(6)}$$

and $$-\frac{G}{r^3} = \frac{1}{4}\cos 3\theta \qquad \text{...(7)}$$

$\because$ $G^2 + 4H^3 < 0$ and $G^2 \geq 0$

$\therefore$ $4H^3 < 0 \Rightarrow H^3 < 0 \Rightarrow H < 0$

From (6), $r^2 = -4H \Rightarrow r = 2\sqrt{-H}$

$\therefore$ from (7), $-\frac{G}{\left(2\sqrt{-H}\right)^3} = \frac{1}{4}\cos 3\theta \Rightarrow \cos 3\theta = -\frac{G}{2H\sqrt{-H}}$

$\therefore$ $$\cos 3\theta = \frac{G}{\sqrt{-4H^3}}$$

This equation gives $\cos\theta$ and so $\cos\theta$ $\cos\left(\theta+\frac{\pi}{3}\right)$ and $\cos\left(\theta+\frac{4\pi}{3}\right)$ are known.

$\therefore$ roots of given equation (4) are

$$2\sqrt{-H}\cos\theta,\ 2\sqrt{-H}\cos\left(\theta+\frac{2\pi}{3}\right) \text{ and } 2\sqrt{-H}\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$\left[\because\ x = rz = 2\sqrt{-H}\,z\right]$$

# ILLUSTRATIVE EXAMPLES

**Example 1.** Use Cardan's method to solve $x^3 - 18x - 35 = 0$.

(G.N.D.U. 2005; Pbi. U. 2005, 2008; P.U. 2006

**Sol.** The given equation is $x^3 - 18x - 35 = 0$

Let $x = u + v$

Cubing both sides, we get,

$$x^3 = u^3 + v^3 + 3uv(u + v)$$

or $\quad x^3 = u^3 + v^3 + 3uvx$

or $\quad x^3 - 3uvx - (u^3 + v^3) = 0$

Comparing (1) and (2), we get,

$$-3uv = -18, \quad \text{or} \quad uv = 6$$

$\therefore \quad u^3 v^3 = 216$

and $u^3 + v^3 = 35$

$\therefore \quad u^3$ and $v^3$ are roots of the equation $t^2 - 35t + 216 = 0$

$\therefore \quad (t - 8)(t - 27) = 0$

$\therefore \quad t = 8, 27$

Let $\quad u^3 = 8, v^3 = 27$

$\therefore \; u = 8, v^3 = 27$

$\therefore \; x = u + v = 2 + 3 = 5$ is one root of given equation (1).

| | | | | |
|---|---|---|---|---|
| 5 | 1 | 0 | – 18 | – 35 |
| | | 5 | 25 | 35 |
| | 1 | 5 | 7 | 0 |

Remaining roots of (1) are given by

$$x^2 + 5x + 7 = 0$$

$$\therefore \quad x = \frac{-5 \pm \sqrt{25 - 28}}{2} = \frac{-5 \pm \sqrt{-3}}{2} = \frac{-5 \pm i\sqrt{3}}{2}$$

$\therefore \quad$ roots of given equation are $5, \dfrac{-5 \pm i\sqrt{3}}{2}$.

**Example 2.** Use Cardan's method to solve $28x^3 - 9x^2 + 1 = 0$.

(Pbi. U. 2005, 2007, 2010, 2011 ; H.P.U. 2000, 2004, 2009; G.N.D.U.

**Sol.** The given equation is $28x^3 - 9x^2 + 1 = 0$

We want to reduce it to the standard form. For that we proceed like this

Put $x = \frac{1}{y}$ [∵ constant term is 1 and coeff. of $x$ is 0]

∴ (1) reduces to

$\frac{28}{y^3} - \frac{9}{y^2} + 1 = 0$ or $y^3 - 9y + 28 = 0$ ...(2)

Let $y = u + v$

∴ $y^3 = u^3 + v^3 + 3uv(u + v)$

or $y^3 = u^3 + v^3 + 3uvy$

or $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(3)

Comparing (2) and (3), we get,

$-3uv = -9 \Rightarrow uv = 3 \Rightarrow u^3v^3 = 27$

and $-(u^3 + v^3) = 28 \Rightarrow u^3 + v^3 = -28$

∴ $u^3, v^3$ are roots of the equation

$t^2 + 28t + 27 = 0$ or $(t + 1)(t + 27) = 0$

∴ $t = -1, -27$

Let $u^3 = -1, v^3 = -27$

∴ $u = -1, v = -3$ are one of the values of $u$ and $v$ respectively.

∴ $y = u + v = -1 - 3 = -4$ is one root of (2)

∴ $x = \frac{1}{y} = -\frac{1}{4}$ is one root of (1)

| | | | | |
|---|---|---|---|---|
| $-\frac{1}{4}$ | 28 | $-9$ | 0 | 1 |
| | | $-7$ | 4 | $-1$ |
| | 28 | $-16$ | 4 | 0 |

∴ remaining roots of (1) are given by

$28x^2 - 16x + 4 = 0$

or $7x^2 - 4x + 1 = 0$

∴ $x = \frac{4 \pm \sqrt{16 - 28}}{14} = \frac{4 \pm \sqrt{-12}}{14} = \frac{4 \pm 2i\sqrt{3}}{14} = \frac{2 \pm i\sqrt{3}}{7}$

∴ roots of given equation are $-\frac{1}{4}, \frac{2 \pm i\sqrt{3}}{7}$.

**Example 3.** Use Cardan's method to solve $x^3 - 6x^2 - 6x - 7 = 0$.

(P.U. 2002, 2012; H.P.U. 2005, 2009, 2010; Pbi. U. 2010, 2011)

**Sol.** The given equation is $x^3 - 6x^2 - 6x - 7 = 0$ ...(1)

Sum of roots = 6

To remove the second term we are to diminish each roots by 2.

| | | | | |
|---|---|---|---|---|
| 2 | 1 | −6 | −6 | −7 |
| | | 2 | −8 | −28 |
| | 1 | −4 | −14 | −35 |
| | | 2 | −4 | |
| | 1 | −2 | −18 | |
| | | 2 | | |
| | 1 | 0 | | |
| | 1 | | | |

Let the transformed equation be in $y$ so that $y = x - 2$

$\therefore$ transformed equation is

$$y^3 - 18y - 35 = 0$$

Let $y = u + v$

Cubing both sides, $y^3 = u^3 + v^3 + 3uv(u + v)$

$\therefore \quad y^3 = u^3 + v^3 + 3uvy$

$$y^3 - 3uvy - (u^3 + v^3) = 0$$

Comparing (3) and (4), we get,

$$-3uv = -18 \quad \Rightarrow \quad uv = 6$$

$\therefore \quad u^3 v^3 = 216, \quad u^3 + v^3 = 35$

$\therefore \quad u^3, v^3$ are the roots of the equation

$$t^2 - 35t + 216 = 0$$

$\therefore \quad (t - 27)(t - 8) = 0$

$\therefore \quad t = 27, 8$

Let $u^3 = 27, \ v^3 = 8$

$\therefore \quad u = 3, v = 2$ are one of the values of $u$ and $v$ respectively.

$\therefore \quad y = u + v = 3 + 2 = 5$

By equation (2), $y = x - 2 \quad \Rightarrow \quad x = y + 2$

$\therefore \quad x = 5 + 2 = 7$ is one of the roots of (1) by synthetic division

| | | | | |
|---|---|---|---|---|
| 7 | 1 | −6 | −6 | −7 |
| | | 7 | 7 | 7 |
| | 1 | 1 | 1 | 0 |

remaining roots of the equation (1) are given by

$$x^2 + x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1 - 4.1.1}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Hence the roots of equation (1) are 7, $\frac{-1 \pm i\sqrt{3}}{2}$.

**Example 4.** Use Cardan's method to solve the following equation :

$2x^3 - 7x^2 + 8x - 3 = 0$

(P.U. 2005 ; H.P.U. 2002)

**Sol.** The given equation is $2x^3 - 7x^2 + 8x - 3 = 0$ ...(1)

Multiplying each root of (1) by 6, we get the new equation is $y$ as

$$2y^3 - 6^1 . 7y^2 + 6^2 . 8y - 6^3 . 3 = 0$$

or $$2y^3 - 42y^2 + 288y - 648 = 0$$

or $$y^3 - 21y^2 + 144y - 324 = 0 \quad ...(2)$$

Here $y = 6x$

Sum of three roots = 21

∴ second term will be missing when we decrease each root by 7.

| | | | | |
|---|---|---|---|---|
| 7 | 1 | −21 | 144 | −324 |
| | | 7 | −98 | 322 |
| | 1 | −14 | 46 | −2 |
| | | 7 | −49 | |
| | 1 | −7 | −3 | |
| | | 7 | | |
| | 1 | 0 | | |
| | 1 | | | |

Let new equation be in $z$ where $z = y - 7 = 6x - 7$

∴ transformed equation is

$$z^3 - 3z - 2 = 0 \quad ...(3)$$

Let $z = u + v$

$\therefore$ $z^3 = u^3 + v^3 + 3uv(u + v)$ $\Rightarrow z^3 = u^3 + v^3 + 3uvz$

$\therefore$ $z^3 - 3uvz - (u^3 + v^3) = 0$ ...(4)

Comparing (3) and (4), we get,

$-3uv = -3$ $\Rightarrow$ $uv = 1$ $\Rightarrow$ $u^3v^3 = 1$

and $-(u^3 + v^3) = -2$ $\Rightarrow$ $u^3 + v^3 = 2$

$\therefore$ $u^3, v^3$ are roots of the equation

$t^2 - 2t + 1 = 0$

$\therefore$ $(t-1)^2 = 0$ $\Rightarrow$ $t = 1, 1$

Let $u^3 = 1, v^3 = 1$

$\therefore$ $u = 1, v = 1$ are one values of $u, v$ respectively.

$\therefore$ $z = u + v = 1 + 1 = 2$ is one root of (3)

Now $z = 6x - 7$ $\Rightarrow$ $2 = 6x - 7$ $\Rightarrow$ $6x = 9$

$\Rightarrow$ $x = \frac{3}{2}$ is one root of (1)

| | | | | |
|---|---|---|---|---|
| $\frac{3}{2}$ | 2 | −7 | 8 | −3 |
| | | 3 | −6 | 3 |
| | 2 | −4 | 2 | 0 |

$\therefore$ remaining roots of (1) are given by

$x^2 - 2x + 1 = 0$

$\therefore$ $(x-1)^2 = 0$ $\Rightarrow$ $x = 1, 1$

$\therefore$ roots of (1) are $\frac{3}{2}, 1, 1$.

**Example 5.** Prove by Cardan's method that the roots of

$$x^3 - 3x + 1 = 0 \text{ are } 2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}.$$ (G.N.D.U. 2004)

**Sol.** The given equation is $x^3 - 3x + 1 = 0$ ...(1)

Let $x = u + v$

$\therefore$ $x^3 = u^3 + v^3 + 3uv(u + v)$

or $x^3 = u^3 + v^3 + 3uvx$

or $x^3 - 3uvx - (u^3 + v^3) = 0$ ...(2)

Comparing (1) and (2), we get,

$-3uv = -3 \Rightarrow uv = 1$

and $-(u^3+v^3) = 1 \Rightarrow u^3+v^3 = -1 \Rightarrow u^3v^3 = 1$

$\therefore$ $u^3, v^3$ are roots of the equation $t^2+t+1=0$

$\therefore$ $t = \dfrac{-1\pm\sqrt{1-4}}{2} = \dfrac{-1\pm\sqrt{-3}}{2} = \dfrac{-1\pm i\sqrt{3}}{2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

Let $u^3 = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$, $v^3 = -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$

Put $-\dfrac{1}{2} = r\cos\theta$ ...(3)

and $\dfrac{\sqrt{3}}{2} = r\sin\theta$ ...(4)

Squaring and adding (3) and (4), we get,

$r^2(\cos^2\theta + \sin^2\theta) = \dfrac{1}{4} + \dfrac{3}{4}$

$\therefore$ $r^2 = 1 \Rightarrow r = 1$

Dividing (4) by (3), $\tan\theta = -\sqrt{3}$

Now $\cos\theta$ is – ve and $\sin\theta$ is + ve [$\because$ of (3) and (4)]

$\therefore$ $\theta$ lies in 2nd quadrant.

$\therefore$ $\tan\theta = -\sqrt{3} \Rightarrow \tan\theta = -\tan\dfrac{\pi}{3} \Rightarrow \tan\theta = \tan\left(\pi - \dfrac{\pi}{3}\right)$

$\Rightarrow$ $\tan\theta = \tan\dfrac{2\pi}{3} \Rightarrow \theta = \dfrac{2\pi}{3}$

$\therefore$ $u^3 = r(\cos\theta + i\sin\theta)$ or $u^3 = r[\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)]$

$\therefore$ $u = r^{\frac{1}{3}}[\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)]^{\frac{1}{3}}$

$\therefore$ $u = (1)^{\frac{1}{3}}\left[\cos\left(\dfrac{2n\pi+\theta}{3}\right) + i\sin\left(\dfrac{2n\pi+\theta}{3}\right)\right]$ where $n = 0, 1, 2$

$\therefore$ $u = \cos\dfrac{2n\pi+\theta}{3} + i\sin\dfrac{2n\pi+\theta}{3}$ where $n = 0, 1, 2.$

$\therefore$ $v = \cos\dfrac{2n\pi+\theta}{3} - i\sin\dfrac{2n\pi+\theta}{3}$, where $n = 0, 1, 2,$

[$\because$ $u^3$ and $u^3$ are conjugate of each other
$\therefore$ $u$ and $u$ are also conjugate of each other]

$\therefore \quad x = u + v = 2\cos\dfrac{2n\pi + \theta}{3}$ where $n = 0, 1, 2.$

$\therefore \quad x = 2\cos\dfrac{\theta}{3},\ 2\cos\dfrac{2\pi+\theta}{3},\ 2\cos\dfrac{4\pi+\theta}{3}$

$= 2\cos\dfrac{\frac{2\pi}{3}}{3},\ 2\cos\dfrac{2\pi+\frac{2\pi}{3}}{3},\ 2\cos\dfrac{4\pi+\frac{2\pi}{3}}{3}$ $\quad \left[\because\ \theta = \dfrac{2\pi}{3}\right]$

$= 2\cos\dfrac{2\pi}{9},\ 2\cos\dfrac{8\pi}{9},\ 2\cos\dfrac{14\pi}{9}.$

**Example 6.** Discuss nature of roots of the equation $x^3 - x^2 - 2x + 1 = 0$

(G.N.D.U. 2007)

**Sol.** The given equation is $x^3 - x^2 - 2x + 1 = 0$ ...(1)

Sum of three roots = 1

$\therefore$ second term will be removed if we decrease each root by $\dfrac{1}{3}$.

| | | | | |
|---|---|---|---|---|
| $\frac{1}{3}$ | 1 | $-1$ | $-2$ | 1 |
| | | $\frac{1}{3}$ | $-\frac{2}{9}$ | $-\frac{20}{27}$ |
| | 1 | $-\frac{2}{3}$ | $-\frac{20}{9}$ | $\frac{7}{27}$ |
| | | $\frac{1}{3}$ | $-\frac{1}{9}$ | |
| | 1 | $-\frac{1}{3}$ | $-\frac{7}{3}$ | |
| | | $\frac{1}{3}$ | | |
| | 1 | 0 | | |
| | 1 | | | |

Let transformed equation be in $y$

$\therefore$ transformed equation is

$$y^3 - \frac{7}{3}y + \frac{7}{27} = 0$$

Comparing it with $y^3 + 3Hy + G = 0$, we get,

$$3H = -\frac{7}{3},\ G = \frac{7}{27}$$

$$\therefore \quad H = -\frac{7}{9},\ G = \frac{7}{27}$$

Now $$G^2 + 4H^3 = \left(\frac{7}{27}\right)^2 + 4\left(-\frac{7}{9}\right)^3 = \frac{49}{729} - \frac{1372}{729} = -\frac{1323}{729} < 0$$

$\therefore$ all the roots of (2) and hence all the roots of (1) are real.

**Example 7.** Discuss the nature of roots of the equation $x^3 + 18x - 35 = 0$.

(G.N.D.U. 2006)

**Sol.** The given equation is $x^3 + 18x - 35 = 0$ ...(1)

Comparing it with $x^3 + 3Hx + G = 0$, we get,

$3H = 18,\ G = -35$

$\therefore \quad H = 6,\ G = -35$

$$\therefore \quad G^2 + 4H^3 = (-35)^2 + 4(6)^3 = 1225 + 864 = 2089$$

Discriminant $= -27(G^2 + 4H^3) = -27(2089) = -56403 < 0$

$\therefore$ all the roots are real and distinct.

# EXERCISE 9 (*a*)

1. Use Cardan's method to solve the following equations for $x$:

(G.N.D.U. 2001; P.U. 2010)

   (*i*) $x^3 - 27x + 54 = 0$ (P.U. 2011)

   (*ii*) $x^3 - 15x - 126 = 0$ (P.U. 2011)

   (*iii*) $x^3 - 30x + 133 = 0$ (P.U. 2010)

   (*iv*) $x^3 - 12x - 65 = 0$

2. Use Cardan's method to solve the following equations for $x$:

   (*i*) $x^3 - 6x - 9 = 0$

   (*ii*) $x^3 - 36x - 91 = 0$

   (*iii*) $x^3 - 21x - 344 = 0$

   (*iv*) $x^3 + 3x - 14 = 0$

   (*v*) $x^3 - 3abx + (a^3 + b^3) = 0$

   (*vi*) $x^3 - 147x + 686 = 0$

3. Solve the following cubic by Cardan's method :

   (*i*) $x^3 - 3x + 3 = 0$

   (*ii*) $x^3 + 9x - 6 = 0$

   (*iii*) $x^3 + 12x + 12 = 0$

   (*iv*) $x^3 - 6x - 6 = 0$

   (*v*) $x^3 - 9x + 28 = 0$

   (*vi*) $x^2 + 72x - 1720 = 0$

   (*vii*) $x^3 - 432x + 3456 = 0$