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SIMPLE HARMONIC MOTION

Art-1. Simple Harmonic Motion

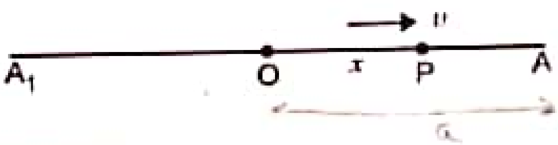
(Pbi. U. 2008, 2011)

A particle is said to execute Simple Harmonic Motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.

Note : Simple Harmonic Motion is briefly written as S.H.M.

Art-2. A particle moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the displacement of the particle from the fixed point. Find the expressions for the velocity and position of the particle at any time. (Pbi. U. 2008, 2011; H.P.U. 2010)

Proof. Let O be the fixed point in the line A_1OA and let P denote the particle after time t from O moving with a velocity v in the positive direction from O to A. Let OP = x, then the acceleration is μx, where μ is constant.



Since the acceleration is in the direction opposite to that in which x increases, the equation of motion of the particle is

$$v\frac{dv}{dx} = -\mu x \qquad ...(1)$$

Separating the variables, $v\,dv = -\mu x\,dx$

$$\text{Integrating, } \frac{v^2}{2} = -\frac{\mu x^2}{2} + c_1 \qquad ...(2)$$

where c_1 is constant of integration.

As P is supposed to be moving in the direction OA and as the acceleration is given to be taking place in the opposite direction, the particle P must come to rest at some point in OA, say at A such that OA = a.

$\therefore \quad v = 0$ when $x = a$

$\therefore \quad$ from (2), $0 = -\frac{1}{2}\mu a^2 + c_1 \quad \Rightarrow \quad c_1 = \frac{1}{2}\mu a^2$

Putting $c_1 = \frac{1}{2}\mu a^2$ in (2), we get,

$$\frac{v^2}{2} = -\mu\frac{x^2}{2} + \frac{1}{2}\mu a^2 \quad \Rightarrow \quad v^2 = \mu(a^2 - x^2)$$

$$\Rightarrow \quad v = \pm\sqrt{\mu}\sqrt{a^2 - x^2} \quad ...(3)$$

This equation gives the value of the velocity v for any displacement x. Since P is moving in the positive direction

$$\therefore \quad v = \sqrt{\mu}\sqrt{a^2 - x^2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \sqrt{\mu}\sqrt{a^2 - x^2}$$

Separating the variables, we get,

$$\frac{1}{\sqrt{a^2 - x^2}}dx = \sqrt{\mu}\,dt$$

Integrating, $\sin^{-1}\frac{x}{a} = \sqrt{\mu}\,t + c_2$

where c_2 is constant of integration. ...(4)

(*i*) If the time t is measured from the instant when P is at O *i.e.* if $x = 0$ when $t = 0$, then from (4),

$$\sin^{-1} 0 = 0 + c_2 \quad \Rightarrow \quad c_2 = 0 \text{ as } \sin^{-1} 0 = 0$$

Putting $c_2 = 0$ in (4), we get,

$$\sin^{-1}\frac{x}{a} = \sqrt{\mu}\,t \quad \Rightarrow \quad \frac{x}{a} = \sin\sqrt{\mu}\,t$$

$$\Rightarrow \quad x = a\sin\sqrt{\mu}\,t \quad ...(5)$$

(*ii*) If the time t is measured from the instant when the particle is at A *i.e.* if $x = a$ when $t = 0$, then $\sin^{-1} 1 = 0 + c_2$

$$\Rightarrow \quad \frac{\pi}{2} = c_2 \text{ as } \sin^{-1} 1 = \frac{\pi}{2}$$

Putting $c_2 = \frac{\pi}{2}$ in (4), we get,

$$\sin^{-1}\frac{x}{a} = \sqrt{\mu}\,t + \frac{\pi}{2} \quad \text{or} \quad \frac{x}{a} = \sin\left(\frac{\pi}{2} + \sqrt{\mu}\,t\right)$$

$$\Rightarrow \quad \frac{x}{a} = \cos\sqrt{\mu}\,t \quad \Rightarrow \quad x = a\cos\sqrt{\mu}\,t \quad ...(6)$$

Equation (5) or (6) gives the position of the particle at any time t.

Note 1. The fixed point O is called the **centre of attraction** or **oscillation** or the **mean position** or the **position of equilibrium.** In this position, the acceleration is zero and so the force acting on the particle is zero.

Note 2. We have $f = -\mu x$. So it is clear that f is negative when x is positive and f is positive when x is negative. Thus, in both the cases, the acceleration and hence the force tries to bring the body back to the equilibrium position. Such a force is called **restoring force.**

Note 3. When the particle is on the left hand side of O, the equation of motion is

$v\frac{dv}{dx}$ = acceleration in the direction of $A_1P = \mu$ OP

$$= \mu(-x) = -\mu x.$$

Hence the same equation that holds on the right a hand side of O, holds also on the left hand side.

Note 4. The points A and A_1, where the velocity is zero, are called the **extreme positions** or **positions of rest.** For these positions, $x = a, -a$.

Note 5. Amplitude The distance $OA = OA_1 = a$ is called the **amplitude** of S.H.M.

Note 6. $v = \pm\sqrt{\mu}\sqrt{a^2 - x^2}$ shows that the particle has equal and opposite velocities at a point according as it is moving in the direction O to A or in the direction A to O.

Cor 1. Maximum and Minimum Velocities

We have $v^2 = \mu(a^2 - x^2)$

$\therefore$ v is maximum when x is least *i.e.* $x = 0$.

$\therefore$ max. velocity $= \sqrt{\mu}\sqrt{a^2 - 0} = \sqrt{\mu}\,a$, which occurs at the mean position

Again v is minimum when x is greatest in magnitude *i.e.* $x = a, -a$

$\therefore$ minimum velocity $= \sqrt{\mu} \times \sqrt{a^2 - a^2} = 0$, which occurs at the extreme position.

$\therefore$ *particle has maximum velocity* $\sqrt{\mu}\,a$ *at O and minimum velocity zero at A or* A_1.

Cor. 2. Maximum and Minimum Accelerations

We have $f = -\mu x$

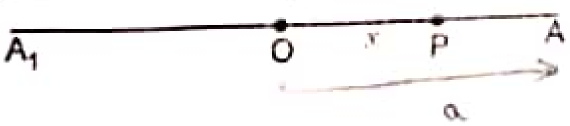
$\therefore$ acceleration (in magnitude) is maximum when $x = a$ or $-a$ and max. acceleration $= \mu a$ and acceleration is minimum when x is minimum *i.e.* $x = 0$ and min. acceleration = 0.

$\therefore$ *particle has max. acceleration* μa at the extreme position A or A_1 and has minimum acceleration zero at the mean position O.

Art-3. Nature of the Simple Harmonic Motion

(P.U. 2011)

Let O be the fixed point in the line A_1OA and let P denote the particle after t from O moving with a velocity v in the position direction from O to A. Let $OP = x$, $OA = a$



$\therefore$ $v = \pm\sqrt{\mu}\sqrt{a^2 - x^2}$

gives the velocity of P in terms of its distance from O. Initially when $x = 0$ at the point O, the velocity is maximum and equal to $\sqrt{\mu}\, a$. As the particle proceeds towards A, then acceleration being towards O, the velocity goes on decreasing as x increases. At A where $x = a$, its vanishes and the particle is, for an instant, at rest. Then owing to the acceleration towards O, the particle moves in the negative direction with a velocity which increases numerically as x decreases and is the greatest at O where it is $-\sqrt{\mu}\, a$. Due to this velocity, the particle proceeds further to the negative side of O, the velocity remaining negative and decreases gradually in magnitude till the particle comes to rest at A_1, where $x = -a$. The acceleration being towards O, the particle the starts and moves towards O with a positive velocity which increases gradually till it is again maximum at O. The same motion is repeated again and again and the particle goes on oscillating indefinitely between A and A_1, the two positions of momentary rest.

Art-4. Periodic Motion

A particle is said to have a periodic motion when it moves in a such a manner that after a certain fixed interval of time, it occupies the same position and moves, in the same direction with the same velocity.

The fixed minimal interval of time for such a motion is called the **period of the motion.**

Art-5. Prove that Simple Harmonic Motion is periodic with period $\frac{2\pi}{\sqrt{\mu}}$.

(Pbi. U. 2010, 2011; P.U. 2007, 2011; H.P.U. 2011)

Proof. Let x be the displacement of the particle and v its velocity at any time t, measured from the centre of oscillation.

$\therefore$ we have

$$x = a \sin \sqrt{\mu}\, t \quad \ldots(1)$$

→ On differentiating, we get

$$\text{and} \quad v = a\sqrt{\mu} \cos \sqrt{\mu}\, t \quad \ldots(2)$$

$$\left[\because \; v = \frac{dx}{dt}\right]$$

We know that $\sin\theta$ and $\cos\theta$ are periodic functions of θ, the period in each case being 2π. That is to say, the value of $\sin\theta$ and $\cos\theta$ is repeated when θ is increased by 2π. Therefore the values of x and v are repeated when $\sqrt{\mu}\, t$ is increased by 2π or when t is increased by $\frac{2\pi}{\sqrt{\mu}}$.

Thus, after every interval of $\frac{2\pi}{\sqrt{\mu}}$, we have the same position and same velocity in the same direction.

Therefore the motion is periodic, the period being $\frac{2\pi}{\sqrt{\mu}}$.

Note 1. Since periodic time $= \frac{2\pi}{\sqrt{\mu}}$

$\therefore$ *Simple Harmonic Motion is periodic and its period is independent of the amplitude.*

Note 2. Frequency is the number of complete oscillations in one second, so that if n denotes the frequency and T the periodic time. (P.U. 2007)

$$nT = 1 \quad \text{or} \quad n = \frac{1}{T} = \frac{\sqrt{\mu}}{2\pi}$$

Frequency = 1 / Periodic Time.

It should be noted that frequency is reciprocal of the periodic time.

Art-6. Important Results of S.H.M.

1. Acceleration $= -\mu x$
2. Acceleration is maximum (in magnitude) at extreme positions *i.e.* when $x = a$, a and maximum acceleration $= \mu a$ (in magnitude)
3. Velocity v is given by $v^2 = \mu(a^2 - x^2)$

   Velocity is maximum at the mean position and maximum velocity (in magnitude) $= \sqrt{\mu}\, a$
4. $x = a \sin \sqrt{\mu}\, t$ when t is measured from the mean position.
5. $x = a \cos \sqrt{\mu}\, t$ when t is measured from the extreme position.
6. Time period $T = \dfrac{2\pi}{\sqrt{\mu}}$
7. Frequency $n = \dfrac{1}{T} = \dfrac{\sqrt{\mu}}{2\pi}$
8. Amplitude $= a$
9. Time from centre to the end point = Time from end point to the centre.

ILLUSTRATIVE EXAMPLES

Example 1. A particle moving with S.H.M. of period 30 seconds travels 15 cm from position of rest in 5 seconds. Find amplitude, maximum velocity and velocity at the end of 5 seconds.

(P.U. 2009 ; G.N.D.U. 2012, 2015)

Sol. Here T = 30 seconds $\Rightarrow \dfrac{2\pi}{\sqrt{\mu}} = 30 \Rightarrow \sqrt{\mu} = \dfrac{\pi}{15}$

Let a be the amplitude.

When $t = 5$, $x = (a - 15)$ cm

$$\therefore \quad x = a \cos \sqrt{\mu}\, t \quad \Rightarrow \quad a - 15 = a \cos\left(\frac{\pi}{15} \times 5\right)$$

Art-7. Motion of a Particle Attached to An Elastic String

A string which stretches under the influence of a force is called an elastic string.

If an elastic string is fixed at one point and pulled within limits at the other, it is found to increase in length. This extension in the string is proportional directly to the product of the tension and the natural length of the string and inversely to the area of the cross-section of the string. If x, l, T, A denote extension, natural length, tension and area of cross-section, then

$$x = \frac{T\,l}{\lambda\,A} \qquad \text{or} \qquad T = \lambda\frac{A\,x}{l}$$

where λ (called the *modulus of elasticity*) is a constant depending on the material of the string.

If the unit of area of cross-section is so chosen that A = 1 unit area, then $T = \lambda\frac{x}{l}$.

This is **Hooke's Law.** It states that *tension of an elastic string is proportional to the extension of the string beyond its natural length.*

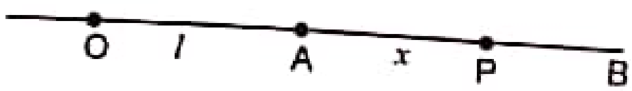
Note. If l is the natural length and l' the extended length of the string, then $x = l' - l$

$$\therefore \quad T = \lambda\,.\,\frac{l' - l}{l}$$

Art-8. Horizontal Elastic String

(P.U. 2006, 2008)

Let one end of an elastic string be fixed to a point O on a smooth horizontal table and let OA = l be its natural length.



If a particle of mass m is attached to the other end and if the particle is displaced along the line OA, a distance AB = b and P be the position of the particle at any subsequent time so that AP = x, then, by Hooke's law, the tension in the string is $\lambda\frac{x}{l}$ which acts in the direction PA and is directed towards A.

The tension of the string being the only force which tends to move the particle, its equation of motion is

$$m\frac{d^2x}{dt^2} = -T \qquad \text{or} \qquad m\frac{d^2x}{dt^2} = -\frac{\lambda}{l}x \qquad \text{or} \qquad \frac{d^2x}{dt^2} = -\frac{\lambda}{l\,m}x$$

which shows that the motion about A is simple harmonic.

Comparing $\frac{d^2x}{dt^2} = -\frac{\lambda}{l\,m}x$ with $\frac{d^2x}{dt^2} = -\mu\,x$, we get,

$\mu = \frac{\lambda}{l\,m}$. Therefore time period $= 2\pi\sqrt{\frac{m\,l}{\lambda}}$

The equation of motion can also be written as

$$v\frac{dv}{dx} = -\frac{\lambda}{l\,m}x \quad \text{or} \quad v\,dv = -\frac{\lambda}{l\,m}x\,dx$$

Integrating, $\frac{v^2}{2} = -\frac{\lambda}{l\,m}\frac{x^2}{2} + c$

At B, $x = b$ and $v = 0$, $\therefore\ 0 = -\frac{\lambda}{l\,m}\frac{b^2}{2} + c$

$$\Rightarrow \quad c = \frac{\lambda b^2}{2\,l\,m}$$

$$\therefore \quad \frac{v^2}{2} = -\frac{\lambda}{2\,l\,m}x^2 + \frac{\lambda b^2}{2\,l\,m} \quad \Rightarrow v^2 = \frac{\lambda}{l\,m}(b^2 - x^2)$$

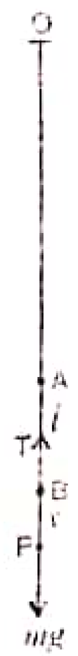
$$\Rightarrow \quad v = -\sqrt{\frac{\lambda}{l\,m}}\sqrt{b^2 - x^2},$$

the sign is taken negative as the particle is moving towards O.

Art-9. Motion of a Particle Attached to a Vertical Elastic String

Prove that the motion of a particle of mass m attached to a horizontal elastic string is simple harmonic with time period $2\pi\sqrt{\frac{m\,l}{\lambda}}$.

Proof. Let a particle of mass m be attached to the lower end of a vertical light elastic string OA of natural length l. The upper end O of the string is kept fixed.



Then the weight $m\,g$ of the particle acting downwards causes extension in string and hence tension in the string in upward direction.

Let the equilibrium in stretched length be attained at position B, where AB = l_1.

By Hooke's law, tension T in the string is given by

$$T = \lambda\left(\frac{OB - OA}{OA}\right) = \lambda\left(\frac{l + l_1 - l}{l}\right) = \lambda\left(\frac{l_1}{l}\right).$$

where λ is the modulus of elasticity of the string

The condition of equilibrium at B is

$$T = m\,g$$

$$\therefore \quad \frac{\lambda\,l_1}{l} = m\,g. \quad \text{...(1)}$$

Take B as origin and the downward direction as positive direction of x-axis.

Let P be the position of the particle at any time t such that BP = x

$$\therefore \quad \text{tension } T' = \lambda\left(\frac{OP - OA}{OA}\right) = \lambda\left(\frac{l + l_1 + x - l}{l}\right) = \lambda\left(\frac{l_1 + x}{l}\right),$$

which acts in upward direction.

$\therefore$ resultant force in

downward direction = $m g - T'$

$$= m g - \lambda\left(\frac{l_1 + x}{l}\right)$$

$$= m g - \lambda\left(\frac{l_1}{l} + \frac{x}{l}\right)$$

$$= m g - \frac{\lambda l_1}{l} - \frac{\lambda x}{l}$$

$$= m g - m g - \frac{\lambda x}{l} \qquad [\because \text{ of } (1)]$$

$$= -\frac{\lambda x}{l}.$$

$\therefore$ equation of motion of the particle is

$$m f = -\frac{\lambda x}{l}$$

or $\quad f = -\left(\frac{\lambda}{m l}\right)x$, which is of the form $f = -\mu x$

$\therefore$ motion of the particle is simple harmonic with $\mu = \frac{\lambda}{m l}$

$\therefore$ time period of complete oscillation $= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\frac{\lambda}{m l}}} = 2\pi\sqrt{\frac{m l}{\lambda}}$.

ILLUSTRATIVE EXAMPLES

Example 1. One end of an elastic string whose modulus of elasticity is λ and whose natural length is l, is tied to a fixed point on a smooth horizontal table and the other end is tied to a mass m lying on the table. The particle is pulled to a distance where the extension of the string becomes a and then let go : describe the character of the motion and show that

(*i*) the string becomes slack after a period of $\frac{1}{2}\pi\sqrt{\frac{m l}{\lambda}}$